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Short Communication

Boundary compatibility condition and rotation in elasticity

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The boundary compatibility condition in elasticity that was missed for more than one century has been formulated. The new condition has completed the Beltrami-Michell formulation in elasticity. The completed formulation can solve stress, displacement, and mixed boundary value problems. The boundary compatibility condition, which is not a rotation, should be imposed on an indeterminate boundary. The use of the new condition is illustrated through the solution of a mixed boundary value problem.

Key words: Indeterminate, Completed, Beltrami, Michell.

INTRODUCTION

The compatibility condition on the boundary of a continuum was missed with the theory of elasticity since the time of St. Venant, about 1860. We have derived the boundary compatibility condition (BCC) from the stationary condition of the variational function of the integrated force method (Patnaik, 1986). The BCC has also been used to solve elasticity problems (Patnaik, 2005). The BCC for a two-dimensional elastic continuum in Cartesian coordinates (x, y) can be written as:

$$\left(\frac{\partial \varepsilon_y}{\partial x} - \frac{\partial \varepsilon_{xy}}{\partial y}\right) a_{vx} + \left(\frac{\partial \varepsilon_x}{\partial y} - \frac{\partial \varepsilon_{xy}}{\partial x}\right) a_{vy} = 0 \qquad (1)$$

Where ε_x, ε_y, and ε_{xy} are the strain components, and a_{vx} and a_{vy} are the direction cosines of the outward normal.

There are critics who still believe the BCC to be wrong. One critic, who is the editor of a prestigious journal, wrote "Your Equation (2c, which is BCC given in equation 1) implies that rotation is uniform on planar surfaces having a normal in the x-direction. This is inconsistent with the observed behavior of elastic bodies and is in contradiction with the existing analysis of elastic bodies that successfully predicts their behavior. In view of this feature of your paper which represents a general deficie ncy, I cannot approve it for publication in the Journal"

Through the medium of this discussion we attempt to explain the misunderstanding that lead to the erroneous conclusion. The compatibility condition forms the backbone of theoretical solid mechanics. If the compatibility condition were set aside, then the theory would degenerate into a few courses in applied mathematics. Yet the compatibility condition was neither fully understood nor utilized.

The BCC when expressed in rotation yields the following condition:

$$\frac{1}{2}\left[\frac{\partial}{\partial y}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) a_{vx} - \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) a_{vy}\right] = a_{vx}\frac{\partial \Omega}{\partial y} - a_{vy}\frac{\partial \Omega}{\partial x} = 0 \qquad (2)$$

Where u and v are displacements, and the rotation

$$\Omega = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

The contention of the critic that BCC implies uniform rotation is consistent with equation (2). However, the flaw pertains to the domain of application.

The BCC is only amenable to derivation from a variational function. The line integral that yields the BCC has the following form:

$$\int_i \left[\left(\frac{\partial \varepsilon_y}{\partial x} - \frac{\partial \varepsilon_{xy}}{\partial y}\right) a_{vx} + \left(\frac{\partial \varepsilon_x}{\partial y} - \frac{\partial \varepsilon_{xy}}{\partial x}\right) a_{vy}\right] (\delta\varphi)\, d = 0 \qquad (3)$$

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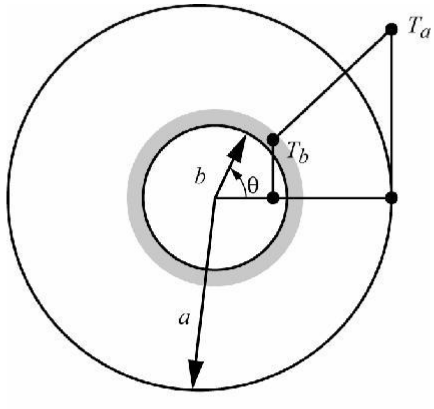

Figure 1. Annular plate subjected to linear temperature distribution.

where φ is the stress function.

Interpretation of the variational term is straight forward. Along a boundary segment, either the variation of the stress function is zero, δφ = 0, and the integral is nonzero or δφ ≠ 0, and the integrand is zero, which is the BCC.

The nonzero δφ implies an indeterminate boundary, while δφ = 0 indicates a determinate boundary. For a two-dimensional elasticity problem, a boundary is one-degree indeterminate when all three stress components are induced. A boundary with prescribed displacements is an indeterminate boundary. A boundary is determinate when one stress component is zero. A boundary segment that can breath (or move) allowing work to be done by the applied traction becomes a determinate boundary. The BCC should be enforced only on an indeterminate boundary segment. The BCC must not be applied to a determinate boundary, and such enforcement can lead to flawed reasoning, as is the case with our esteemed critic. A three-dimensional elasticity problem can be three-degree indeterminate because it has six stress and three displacement components. Its boundary can be one-, two-, or three-degree indeterminate. As a rule, we should impose boundary compatibility conditions on indeterminate boundaries. One BCC for a one-degree indeterminate boundary, two for a two- degree indeterminate boundary, and so forth. The use of the term "indeterminate" may not be very popular in elasticity, but authors have used it (Sada, 1974).

Consider next the field compatibility condition. When expressed in terms of rotation, it transforms into a null condition:

$$\frac{\partial^2 \Omega}{\partial x \partial y} - \frac{\partial^2 \Omega}{\partial x \partial y} = 0 .$$

It would be ridiculous to conclude the nonexistence of the field compatibility condition because it became a null condition in rotation. In other words, expressing BCC in rotation for an erroneous criticism is not justified. The new boundary compatibility condition has completed the Beltrami-Michell formulation in elasticity. The completed formulation can solve stress, displacement, and mixed boundary value problems in elasticity (Patnaik, 2005).

Love's (Love, 1944) inference is quoted below. Earlier, it was valid only for a stress boundary value problem, but now with the BCC, it is valid for all three class of boundary value problems in elasticity:

"It is possible by taking account of these [field and boundary compatibility] relations to obtain a complete system of equations [or, Completed Beltrami Michell Formulation] which must be satisfied by stress components, and thus the way is open for a direct determination of stress without the intermediate steps of forming and solving differential equations to determine the components of displacements."

We believe the theory of elasticity remained incomplete for more than one century. The deficiency pertained to the strain formulation (or the compatibility condition). A veteran researcher should not be surprised over a deficiency in the strain formulation because some of the formulae and equations of the solid mechanics discipline were not completed in the first attempt, but were perfected eventually. For example, perfecting the flexure formulae required more than one century between Galileo, Bernoulli, and Coulomb. Saint-Venant completed the shear stress formula that was initiated by Navier. Cauchy formulated the stress equilibrium equation that was also attempted by Navier in terms of displacement, but it contained only a single material constant instead of two, or the rari-constant theory with one quarter for Poisson's ratio. Green subsequently resolved the misconception. We have completed the strain formulation that was initiated by Saint-Venant in 1860. The new information should be utilized to improve the theory of elasticity.

Illustrative Example

The use of the boundary compatibility condition is illustrated through the solution of a radially symmetrical annular plate. It is made of an isotropic material with Young's modulus (*E*), Poisson's ratio (υ), and coefficient of thermal expansion (α). It has thickness (*h*) (considered unity) with outer and inner radii of (*a* and *b*), respectively, as shown in figure 1. The inner boundary is fully restrained while the outer boundary is free to expand. It is subjected to a linear temperature distribution with values (T_a and T_b) at ($r = a$ and $r = b$), respectively.

$$T = T_b + \frac{(T_a - T_b)}{(a - b)}(r - b) \qquad (4)$$

The response variables of the problem consists of two stresses $(\sigma_r \ and \ \sigma_\theta, n = 2)$ and one displacement (u, m

= 1). In the field, the problem is one degree indeterminate (r = n – m = 1). With one zero stress component, $(\sigma_r = 0)$, the free outer boundary at (r = a) is determinate. Boundary compatibility condition should not be imposed on the outer determinate boundary. The inner restrained boundary at (r = b) is indeterminate because it has two stresses

$$(\sigma_r \neq 0, \sigma_\theta \neq 0)$$

and one BCC should be imposed.

The equations of the completed Beltrami Michell form - ulation for the annular plate subjected to a thermal load are given below (Patnaik SN and Hopkins DA 2005).

Field equations: In the field, there is an equilibrium equation and a compatibility condition.

$$\frac{\partial \sigma_r}{\partial r} + \frac{(\sigma_r - \sigma_\theta)}{r} = 0$$

$$\frac{\partial}{\partial r}(\sigma_\theta - \upsilon\sigma_r) + \frac{(1+\upsilon)}{r}(\sigma_\theta - \sigma_r) = -\alpha E \frac{dT}{dr} \qquad (5)$$

The outer free determinate boundary has a traction condition.

$$\sigma_r = 0 \text{ at } r = a \qquad (6a)$$

The inner clamped indeterminate boundary has a boundary compatibility condition.

$$\sigma_\theta - \upsilon\sigma_r = -\alpha E T \text{ at } r = b \qquad (6b)$$

The clamped boundary has a displacement condition, which is not used in the calculation of the stress state. It is used to back-calculate displacement from stress.

$$u = 0 \text{ at } r = b \qquad (7)$$

The stresses are obtained as solution to the equations (5, 6a, 6b).

$$\sigma_r = \frac{E\alpha(a-r)}{3r^2(a-b)\left\{a^2+b^2+\upsilon(a^2-b^2)\right\}} \times \left\{ T_a\left[r^2\left(a^2+b^2+\upsilon(a^2-b^2)\right) + rb^2\left(a(1-\upsilon)-b(2-\upsilon)\right) + ab^2\left(a(1-\upsilon)-b(2-\upsilon)\right)\right] + T_b\left[-r^2\left(a^2+b^2+\upsilon(a^2-b^2)\right) + rb^2\left(a(2+\upsilon)-b(1+\upsilon)\right) + ab^2\left(-b(1+\upsilon)+a(2+\upsilon)\right)\right]\right\} \qquad (8a)$$

$$\sigma_\theta = \frac{E\alpha}{3r^2(b-a)\left\{a^2+b^2+\upsilon(a^2-b^2)\right\}} \times \left\{ T_a\left[2r^3\left(a^2+b^2+\upsilon(a^2-b^2)\right) - r^2\left(a^3+2b^3+\upsilon(a^3-b^3)\right) + a^2b^2\left(a(1-\upsilon)-b(2-\upsilon)\right)\right] + T_b\left[-2r^3\left(a^2+b^2+\upsilon(a^2-b^2)\right) + r^2\left(a^3-b^3+3ab^2+\upsilon(a^3-b^3)\right) + a^2b^2\left(a(2+\upsilon)-b(1+\upsilon)\right)\right]\right\} \qquad (8b)$$

The displacement function was back-calculated by integrating the stresses. The displacement boundary condition (u = 0 at r = b) was used to evaluate the integration constant. The displacement function is:

$$u = \frac{\alpha(1+\upsilon)(b-r)}{3r^2(b-a)\left(a^2+b^2+\upsilon(a^2-b^2)\right)} \times \left\{ T_a\left[r^2\left(a^2+b^2+\upsilon(a^2-b^2)\right) + ra^2\left(a-2b-\upsilon(a-b)\right) + a^2b\left(a(1-\upsilon)-b(2-\upsilon)\right)\right] + T_b\left[-r^2\left(a^2+b^2+\upsilon(a^2-b^2)\right) + ra^2\left(2a-b+\upsilon(a-b)\right) + a^2b\left(a(2+\upsilon)-b(1+\upsilon)\right)\right]\right\} \qquad (9)$$

The numerical values of the response parameters for (T_a = 100 °C, T_b = 50 °C, $\alpha = 12\times10^{-6}$ /°C, E $=30\times10^{6}$ psi, v = 0.3, a = 20 in. and b = 10 in.) are

(1) at $r = a$: σ_r = 0 ksi, σ_θ = –17.5 ksi, and u = 0.012 in.

(2) at $r = b$: σ_r = 14.2 ksi, σ_θ = –13.7 ksi, and u = 0 in.

The sum of the stresses ($\sigma_r + \sigma_\theta$ = 18508 – 1800 r) has a linear variation with respect to the r-coordinate because of a similar distribution of temperature (see Eq. (4)).

CONCLUDING REMARKS

For an indeterminate boundary, the number of stress components exceeds the number of displacement components. For a determinate boundary, the number of stress components is equal to the number of displacement components. The boundary compatibility conditions should be imposed only on an indeterminate boundary. They should not be imposed on a determinate boundary. The boundary compatibility condition has completed the Beltrami Michell formulation for an elastic continuum with stress and displacement boundary conditions. Stress, the primary variables of the method, is calculated from the equilibrium equations and compatibility conditions. Displacement can be back-calculated from the stress state.

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