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Full Length Research Paper

A directory of absolutely aliased signals

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In this paper, we present a new method for signal reconstruction from multiple sets of samples with unknown offsets which can be written as a set of polynomial equations in both the unknown signal coefficients and the offsets. The solution can then be computed using Groebner bases. In any practical setting, the samples are corrupted by noise, and then there is no algebraic solution. Thus, the next step is to address this noisy version of the problem, and to show how a good approximation can be obtained from multiple Groebner bases for subsets of samples. This provides us with an elegant solution method for the initial nonlinear problem. We show two examples for the reconstruction of polynomial signals and Fourier series.

Key words: Algebraic geometry, deconvolution, exact deconvolution, finite response, Groebner Basis, multichannel, multidimensional, multivariate.

INTRODUCTION

Groebner basis theory is a very powerful tool from algebraic geometry. The theory was originally introduced by Buchberger (1965), and can be found in some very good text books, like for example the book by Cox et al. (1996), as well as in many free (Macaulay2, Singular) and commercial (Magma, Maple, Mathematica) software packages. Groebner bases have also found their way into many applications in signal processing and system theory (Buchberger, 2001a, b). Examples can be found in filter bank design (Charoenlarpnopparut, 2000; Faugere et al., 1998; Kalker et al., 1995), multichannel deconvolution (Unser and Zerubia, 1997), or motion estimation (Holt et al., 1996). In this last paper, Holt et al. (1996) use algebraic geometry to determine the number of solutions and uniqueness for certain problems in three-dimensional motion estimation. They analyze the 3D motion of a rigid link moving in a plane where one endpoint is known, and the extraction of 3D motion from 2D optical flow information. We will consider here shifts of one dimensional signal, which can be extended to global planar shifts of images in the image plane.

This paper is structured as follows: The multichannel sampling problem with unknown offsets is formulated mathematically as a set of polynomial equations in Section 2. Section 3 gives the general multichannel sampling. Section 4 describe the multichannel sampling as a set of polynomial equations, gives an overview of Groebner basis theory, and more particularly the main ideas that we will use for our reconstruction problem. Groebner bases are then applied to the multichannel sampling problem in Section 5. Section 6 gives the ideal membership problem, Section7, describe the Buchberger's algorithm. Solution of polynomial equations and given numerical example in Section 8. Finally, Section9 concludes this paper.

Problem setup

Let us consider a finite dimensional Hilbert space H, for which we have a basis $B = \{ {}_{l}(t) \}_{l=0,\ldots,L-1}$. That is, H = span $\{ {}_{l}(t) \}_{l=0,\ldots,L-1}$. For simplicity, let us consider the functions $\{ {}_{l}(t) \}$ defined on the interval [0, 1]. For periodic functions, we will assume the period to be 1, such that we consider one period.

An arbitrary continuous-time signal f(t) from this space can be expressed as

$$f(t) = \sum_{l=0}^{L-1} {}_{l}\ {}_{l}(t) \qquad (1)$$

Where ${}_{l}$ is the l-th expansion coefficient of f(t) in the L-dimensional basis B . Possible examples of spaces with

general non−uniform sampling with known locations	general non−uniform sampling with unknown locations	uniform sampling
multichannel sampling with known offsets	**multichannel sampling with unknown offsets**	

Figure 1. Classification of sampling methods. Sampling methods can be divided into uniform and non-uniform methods. The non-uniform sampling methods can be subdivided depending on whether the locations are known and whether the samples are grouped in uniform sets with only unknown offsets. In super-resolution, we are interested in multichannel sampling methods with unknown offsets.

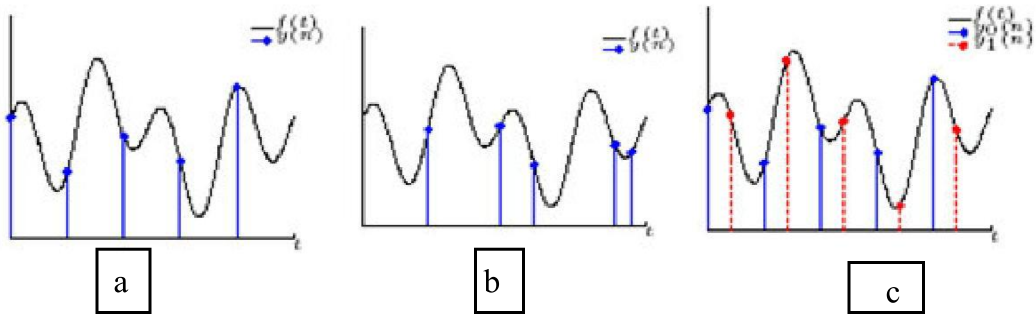

Figure 2. Illustration of the different sampling methods. (a) Uniform sampling. (b) Nonuniform sampling. (c) Multichannel sampling.

associated bases include truncated Fourier series, wavelets, splines, etc.

Assume now that we sample f(t) at times τ, resulting in the sampled signal x(n):

$$(\tau(n))\ x(n) = f \qquad (2)$$

Sampling methods can be classified into different categories, according to the way the sampling times τ are chosen (Figure 1). A recent overview of sampling methods is given by Unser (2000) . If the samples are taken uniformly, at a constant rate N, we have uniform sampling (Figure 2(a)):

$$\tau = (0\ \ \frac{1}{N}\ \ \frac{2}{NN} ... \frac{N-1}{}\ ) \qquad (3)$$

The sampled signal can be written as

$$\text{for } 0 \qquad n < N.\ f\frac{n}{N} \quad x(n) = \qquad (4)$$

This is the standard sampling setup as it is most often used, and as it is also presented in the sampling theorem by Whittaker (1915), and Shannon (1948).

When the samples are not chosen uniformly, the sampling methods are logically called non-uniform (Figure 2b). Among the non-uniform sampling methods, a distinction needs to be made between methods where the sampling instants τ are known (Almansa, 2002; Marvasti, 2001; Strohmer, 1997) and other methods where the sampling locations are unknown. If the sampling locations are unknown and completely arbitrary, the problem cannot be solved. This can be shown using a simple counting argument. Assume that the signal to be reconstructed has L unknown parameters. For every additional sample, there is also an additional unknown (its location). Therefore the number of unknowns is always larger than the number of measurements, and this problem is unsolvable. However, for discrete signals, where the sampling locations can only take a finite number of values, a combinatorial solution can be found, as described by Marziliano and Vetterli (2000).

General multichannel sampling

An important subset of the non- uniform sampling methods is formed by multi channel sampling methods. In divided into M sets of uniformly spaced samples τ_m .

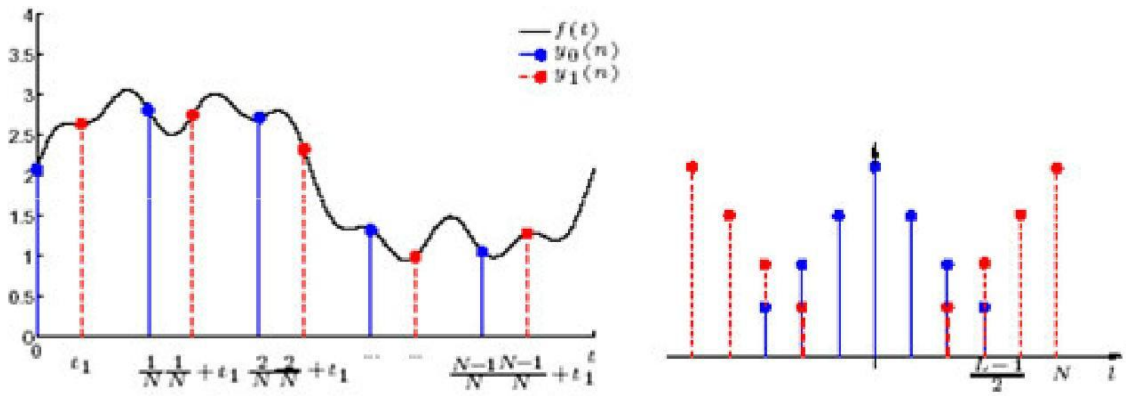

Figure 3. Illustration of the different variables with M = 2 and a Fourier basis. (a).Time domain representation of the signal f(t) and its sets of samples y0(n) (—) and $x_1(n)$ (– –). (b) Frequency domain representation of the absolute values of the signal spectrum (—) and its aliased copies after sampling (– –).

these methods, the set of sampling instants τ can be Each of the sets of samples τ_m is uniform, but the different sets have an arbitrary number of sets t_m (Figure 2c). Note that t_m is expressed in samples. In such a case, the m-th set of samples can be written as

$$x_m(n) = f(\tau_m) = f\frac{n+t_m}{N} \quad \sum_{l=0}^{L-1} {}_{l}\,{}_{l}\frac{n+t_m}{N}, \text{ for } 0 \le n < N \quad (5)$$

Papoulis described a solution for multichannel sampling with known sampling locations (Papoulis, 1977). He showed that a band limited signal can be perfectly reconstructed from M sets of samples that are uniformly sampled at 1/M the Nyquist sampling rate. This result was extended by Unser and Zerubia in their generalized sampling approach [Unser and Zerubia, 1997a, b) . The problem with multiple sets and unknown sampling locations was solved for discrete-time signals by Marziliano and Vetterli (2000). They developed a combinatorial method to compute the discrete offsets between the different sets of samples. In this work, we will study the continuous-time case: multichannel sampling with unknown, real-valued offsets t_m.

Using vector notation, (5) can be written more compactly as

$$x_m = {}_{t_m} \quad (6)$$

In this equation, x_m is the N × 1 vector containing the m-th uniform set of samples, and is the L×1 vector of expansion coefficients. The N ×L matrix ${}_{t_m}$ contains the sampled basis functions that are uniformly sampled with an offset t_m.

Now, all the sets of samples y_m are combined into a single vector y and similarly the basis matrices. ${}_{t_m}$ are combined into ${}_t$, with t = (t_0 t_1… t_{M-1}) denoting the offset vector. This can be written as;

$$y = \begin{matrix} x_0 \\ x_1 \\ \vdots \\ X_{M-1} \end{matrix} = \begin{matrix} t \\ t_1 \\ \vdots \\ t_{M-1} \end{matrix} = {}_t \quad (7)$$

The matrix ${}_t$ has size MN × L. Assuming that the total number of samples is larger than or equal to the number of expansion coefficients, or MN L, this set of equations is in general well-or over-determined if t is known. If, additionally,

$$MN \;\; L+M-1, \quad (8)$$

the number of equations is also larger than or equal to the number of unknowns (L expansion coefficients and M-1 offsets), and it should be possible to remove the uncertainty of the unknown offsets. As we will show in the next sections, these additional equations allow us in general to compute the unknown offsets. Note that the challenging part of the problem is that it is a nonlinear problem in the unknown offsets and expansion coeffi-cients. In summary, the most important variables in this reconstruction problem are listed here (Figure 3). N: the number of samples in each set x_m. x_m: the length N vector of the m-th set of samples. L: the number of unknown expansion coefficients. : The length L vector of the expansion coefficients ${}_l$ to be reconstructed. M: the number of sets of samples. t: the length M vector of the offsets t_m between the different sets of samples.

The unknown variables are the expansion coefficients and the offsets t. We assume that all the other varia-bles are known. This is evident for the sets of samples x_m, the number of samples per set N and the number of sets M, as they form the input of the problem. We will also require that the number of signal coefficients L, or at least an estimate for L, is available.

Multichannel sampling as a set of polynomial equations

Using the setup from section 3, we can write the sample vector as in (7):

$$y = {}_t \,. \qquad (9)$$

We illustrate with an example for second order poly-nomials like in Example (1). Now we fill in the signal parameters for a specific signal.

In the above example, we obtain a set of nonlinear polynomial equations. The equations are linear in the unknown signal coefficients . Thanks to the specific cho-ice of a polynomial basis $\{ {}_l (t) = t^l \}$, the equations are polynomials in the offsets t. Note that for an arbitrary basis $\{ {}_l (t) \}$, this is not valid. However, for certain ba-ses, we can rewrite the equations (7) as a set of poly-nomial equations using a change of variables. This is possible when the basis is a set of functions ${}_l (t) = h(t)^l$, with h(t) an invertible function.

Probably the most important and practically useful example of such a basis is when $h(t) = e^{j2\pi t}$, that is, the Fourier series. In fact, consider the case of a complex signal of the form

$$f(t) = \sum_{l=-k}^{k} {}_l \, {}_l (t) \qquad (10)$$

With ${}_l (t) = e^{j2\pi lt}$. The samples are given by

$$x_m(n) = f\frac{n+t_m}{N} = {}_l \mathop{W}_{t=-k}^{k}{}^{nl}\, e^{\frac{j2\pi l t_m}{N}} \quad \text{for } 0 \le n < N, \qquad (11)$$

With $W = e^{\frac{j2}{N}}$. By setting $z_m = e^{\frac{j2\pi t_m}{N}}$, we obtain

$$x_m(n) = f\frac{n+t_m}{N} = {}_l \mathop{W}_{t=-k}^{k}{}^{nl}\, z_m^{l} \qquad (12)$$

We multiply (12) with z_m^k to eliminate negative expo-nents:

$$z_m^k\, x_m(n) = z_m^k\, f\frac{n+t_m}{N} = {}_l \mathop{W}_{t=-k}^{k}{}^{nl}\, z_m^{l+k}. \qquad (13)$$

For each sample, this can be rewritten as a polynomial constraint

$$P_{nN+m} = {}_l \mathop{W}_{t=-k}^{k}{}^{nl}\, z_m^{l+k} - z_m^k\, x_m(n) = 0. \qquad (14)$$

In this equation, the unknowns are the signal para-meters ${}_l$ and the offset dependent variables z_m. As in Example 1, the equations are linear in the signal para-meters and polynomial in the offset variables z_m. We will now introduce Groebner bases and Buchberger's algo-rithm, which provide an elegant method to solve such a set of polynomial equations.

Groebner bases

We present here the main results related to our multi-channel sampling problem and we refer to Cox et al. (1996) and Buchberger (Buchberger, 2001a b) for a com-plete presentation of algebraic geometry and Groebner bases. This section is intended as a quick introduction and overview of key results that are necessary to our solution method. It can be skipped by readers familiar with Groebner bases.

Affine varieties and ideals

We consider polynomials in the n complex variables, $y_0,\ldots, y_{n-1}$. A polynomial p can then be written compactly as

$$p = \sum_d a_d\, y^d \qquad a_d \in \mathbb{C} \qquad (15)$$

Where the sum is over a finite number of n-tuples $d = (d_0,\ldots,d_{n-1})$ and x^d is a compact notation for $y_0^{d_0},\ldots, y_{n-1}^{d_{n-1}}$.

Each term of the sum in (15) is called a monomial. In the following, we will denote $\mathbb{C}[y_0,\ldots, y_{n-1}]$ the set of (complex) polynomials in the variables $y_0,\ldots, y_{n-1}$.

The basic objects of algebraic geometry are affine va-rieties:

Definition 5.1.1 (Affine Variety): Consider the polynomials $p_0,\ldots,p_{s-1}$ in the n variables $y_0,\ldots,y_{n-1} \in \mathbb{C}$ .Then we set,

$V(p_0,\ldots,p_{s-1}) = \{(c_0,\ldots,c_{n-1}) \in \mathbb{C}^n : p_i(c_0,\ldots,c_{n-1}) = 0, \forall\, 0 \le i \pi s\}$. (16)

We call $V(p_0,\ldots,p_{s-1})$ the affine variety defined by $p_0,\ldots,p_{s-1}$. The elements of an affine variety are the points for which the polynomials $p_0,\ldots,p_{s-1}$ are all zero.
The determination of the affine variety is trivial in the linear case, since the polynomial p_i has the simple form

$p_i(y_0,\ldots, y_{n-1}) = a_{i0}x_0 + \ldots + a_{i(n-1)}y_{n-1} + b_i$, I = 0, . . . , s-1 (17)

and the points of the variety $V(p_0,\ldots,p_{s-1})$ are those that satisfy the system

$$Ay + b = 0, \qquad (18)$$

With $\{A\}_{i,j} = a_{ij}$ and $b = (b_0,\ldots,b_{s-1})^T$. The solution can be easily computed by using Gaussian elimination. Recall that Gaussian elimination consists in computing linear combinations of the rows of (18) in order to remove progressively the variables. The method is based on a certain ordering of the variables. For example, with the ordering $x_0, y_1,\ldots,y_{n-1}$, we obtain a system

$$\bar{A}\, y + \bar{b} = 0 \qquad (19)$$

The i-th row of $\bar{A}$ has the form

$$(0 \quad \ldots \quad 0 \quad \bar{a}_{ij_i} \quad \bar{a}_{ij_{i+1}} \quad \bar{a}_{ij_n}) \qquad (20)$$

The leading zeros in each row correspond to the positions of the variables that have been eliminated from the previous equations. Therefore, we have (possibly with an initial reordering of the equations)

$$j_1 < j_2 < \ldots < j_l < n, \qquad (21)$$

and the rows l+1 to s are all zero. That is, at least one of the variables is eliminated at each step (and possibly more than one). Note that, after the l-th equation, all the variables are eliminated. If $\bar{b}_l = \ldots = \bar{b}_{s-1} = 0$, $\text{rank}(\bar{A}|\bar{b}) = \text{rank}(\bar{A}) = l$ and the system admits a solution. The solution of the system is obtained by back substitution.

The procedure of Gaussian elimination can be extended to the case of polynomial equations. This extension is known as Buchberger's algorithm and the set of equations obtained after elimination is called a Groebner basis. In order to give an overview of the algorithm, we recall the theoretical background and show the analogy with Gaussian elimination. We refer to the bibliography for the details and formal proofs.

As in the linear case, we need to define an ordering of the terms of (15), that is, the monomials of $y_0, \ldots, y_{n-1}$. Since the variables may appear with different exponents, there are different ways to order monomials according to the variables and the exponents. A common choice is lexicographic (lex) ordering.

Definition 5.1.2 (Lexicographic ordering): Let $d = (d_0,\ldots,d_{n-1})$ and $\bar{d} = (\bar{d}_0, \ldots, \bar{d}_{n-1})$ be two n-tuples representing positive integer exponents of the monomials y^d, $y^{\bar{d}}$. We say that $d >_{lex} \bar{d}$ if, in the vector difference $d - \bar{d} \in Z^n$, the left- most nonzero entry is positive. We will write $y^d >_{lex} y^{\bar{d}}$ if $d >_{lex} \bar{d}$.

Note that, next to the type of ordering, we also need to define the order between the different variables. In the following, we will assume that the terms of each polynomial are ordered in descending order according to lex ordering, and with $y_0 > y_1 > \ldots > y_{n-1}$. We define the multidegree of a polynomial p, multideg (p) as the largest exponent of the monomials of p according to the lex ordering. We call leading term, LT(p) the term of p with the largest exponent.

The total degree of a polynomial is defined as the maximum sum of the exponent vectors d of its terms.

Definition 5.1.3 (Ideal): A subset $I \in C[x_0,\ldots,x_{n-1}]$ is ideal if it satisfies: 1. $0 \in I$. 2. If $p, q \in I$, then $p + q \in I$. 3. If $p \in I$ and $a \in C[y_0,\ldots,y_{n-1}]$, then $ap \in I$.

If $p_0,\ldots, p_{s-1}$ are polynomials, then we set

$$I = (p_0,\ldots,p_{s-1}) = \sum_{i=0}^{s-1} a_i p_i : a_i \in C[y_0, \ldots, y_{n-1}] \qquad (22)$$

We call I the ideal generated by $p_0,\ldots,p_{s-1}$.

The ideal membership problem

A key problem in algebra is to determine whether a given element p of a ring belongs to a given ideal I or not. In terms of polynomials, the problem is equivalent to testing if a given polynomial p can be written as a linear combination of the polynomial generators of I, $p_0,\ldots,p_{s-1}$, using polynomial coefficients $a_0,\ldots,a_{s-1}$. Such a problem is known as the Ideal Membership Problem.

If we think of an ideal generated by a single polynomial in one variable, the problem has a simple solution. In fact, we can apply the algorithm of polynomial division and write p as:

$$p = a_0 p_0 + r. \quad (23)$$

The quotient a_0 and the remainder r are uniquely determined under the condition that deg(r) < deg(p_0). In this case, the ideal membership problem has a simple solution: if r = 0, p belongs to $\langle p_0 \rangle$, otherwise not.

In the case of multiple polynomials in multiple variables, we can extend the algorithm of polynomial division. The goal is to write p as

$$p = a_0 p_0 + . . . + a_{s-1} p_{s-1} + r. \quad (24)$$

The division algorithm consists in considering the monomials of p in decreasing order. For each monomial, if the leading term of one of the pi's is a divisor, then the corresponding quotient a_i is updated together with the remaining monomials of p. Otherwise, the monomial is moved to the remainder r.

Theorem 6.1 Fix a monomial order and let P = (p_0,…,p_{s-1}) be an ordered s-tuple of polynomials in y_0, . . . ,y_{n-1}. Then every polynomial p can be written as in (24), where either r = 0 or r is a linear combination of monomials, none of which is divisible by any of LT(p_0),…,LT(p_{s-1}). Further-more, we have

$$\text{multideg}(p) \ \text{multideg}(a_i p_i), \ i = 0,\dots,s-1. \quad (25)$$

A crucial point of the algorithm is that the result of the division depends on the order that we consider for the divisors p_0,…,p_{s-1}.

Definition 6.2 (Groebner basis). Let G = {g_0, . . . , g_{u-1}} be a basis for the ideal I. If for all p ∈ I. the remainder of the division p^G = 0 then G is called a Groebner basis for I.

Theorem 6.3. (Hilbert Basis Theorem). Every ideal I of the ring of polynomials of n variables has a finite generating set. That is, I = (g_0,…,g_{u-1}) for some g_0,…,g_{u-1} I. In particular, it is always possible to choose g_0,…,g_{u-1} so that they form a Groebner basis.

Buchberger's algorithm

The key step of Gaussian elimination was to combine two rows of the matrix (that is, two equations) in order to cancel the entry corresponding to the variable of highest order. This concept is extended to polynomials by introducing S-polynomials.

Definition 7.1 (S-polynomial). Let a_0, a_1 be two non-zero polynomials in y_0,…,y_{n-1}. If multideg (a_0) = d and multideg (a_1) = d , then let d = (d ,…,d $_{n-1}$), where d $_i$ = max(d_i, d $_i$). The S-polynomial of a_0 and a_1 is defined s the linear combination.

$$S(a_0, a_1) = \frac{y^{d''}}{LT(a_0)} a_0 - \frac{y^{d''}}{LT(a_1)} a_1 \quad (26)$$

Theorem 7.2 Let I be a polynomial ideal. Then a basis G = {g_0, . x . . , g_{u-1}} is a Groebner basis for I if and only if, for all pairs i j, the remainder on division of S(g_i,g_j) by G (listed in some order) is zero.

There is a main difference between the linear and the polynomial case when we combine equations. In the linear case, if we combine a_0 and a_1 we obtain an equation of the form

$$h = c a_0 + d a_1, \ c,d \in C. \quad (27)$$

and this equation can be used to replace a_0 or a_1, i.e.

$$\langle a_0, a_1 \rangle = \langle a_0, h \rangle = \langle h, a_1 \rangle. \quad (28)$$

In the polynomial case, equations are combined using polynomial coefficients, that is, the terms a and b are polynomials in the variables y_0,…,y_{n-1}. Since the set of polynomials has the structure of a ring, it is not always to find an inverse of the coefficients. This means that, for example, it is not always possible to compute a_1 from a_0 and h. For this reason, to construct a Groebner basis, one has to increase initially the number of elements of the basis. Such an extension ends when the conditions given by Theorem 7.2 are satisfied. This algorithm is due to Buchberger and is given in the following algorithm.

Algorithm 1: Buchberger's algorithm for the computation of a Groebner basis.

Let I = (a_0,…,a_{s-1}) 0 be a polynomial ideal. Then a Groebner basis for I can be constructed in a finite number of steps by the following algorithm:

Input: A = (a_0,…,a_{s-1})

Output: a Groebner basis G = (g_0,…,g_{u-1}) for I, with A ⊆ G

G := A

Repeat

G := G

For each pair (a, q), a q in G do

S:= S(a, q)^G

If S 0 then G: = G Y S

until G = G .

Algorithm 1 is not a very practical way to compute a Groebner basis.

Several improvements are possible. Moreover, Groebner bases computed in this way are often bigger than necessary. For this reason, unneeded generators are eliminated by using Theorem 7.2 or similar tests.

Solution of polynomial equations

We can now show that a Groebner basis corresponding to a system of polynomial equations and built using lex ordering simplifies the system and allows to compute the solution by back substitution. Remember that we defined the ideal I as the set of all polynomials that can be derived from the initial set using polynomial coefficients. We can also define the elimination ideal I_k as the set of all polynomials that can be deduced from the original system and contain only the variables $y_k,\ldots,y_{n-1}$,

$$I_k = I \quad C[y_k,\ldots,y_{n-1}]. \qquad (29)$$

If we can find a basis for each one of the sets I_k, k =1,…,n-1, we can determine the solutions of the original system using back substitution. In fact, we clearly have that for any k 1, $I_{k+1} \subseteq I_k$. Therefore, if we have a solution of the system of equations associated to I_{k+1}, we can extend it to the system

Multichannel sampling using groebner bases

Instead of computing I_k we compute y_k. This can be done by computing the zeros of a polynomial in the variable y_k. An important property of Groebner bases is that they solve easily the problem of determining the ideals I_k, k = 1,…,, n-1. Namely, the Groebner bases of all the ideals I_k, k = 1,…,n-1 can be determined from the Groebner basis of I.

Theorem 8.2 (Elimination Theorem). Let $I \subset [y_0,\ldots,y_{n-1}]$ be an ideal and let G be a Groebner basis of I with respect to lex order where $x_0 > y_1 > \ldots > y_{n-1}$. Then, for every 1 k < n, the set

$$G_k = G \quad C[y_k,\ldots,yx_{n-1}] \qquad (30)$$

is a Groebner basis of the k-th elimination ideal I_k.

Using this theorem, we can compute the different variables from a Groebner basis using back substitution. To summarize, we can solve a set of polynomial equations in multiple variables as follows. First, we compute a Groebner basis for the ideal corresponding to the set of equations using Buchberger's algorithm. The solution can then be obtained from this Groebner basis using back substitution.

Multichannel sampling using groebner bases

We can now use Groebner bases and Buchberger's algorithm to solve the equations from (1). After a possible change of variables to write the equations in polynomial form, we can directly apply Buchberger's algorithm. This result in a Groebner basis for the ideal is defined by the set of equations. The signal parameters can then be easily extracted from this Groebner basis using the elimination theorem. This is summarized in Algorithm 8.4.

We will illustrate this algorithm with two examples for polynomial signals and signals described by Fourier series.

Algorithm 8.4: Algorithm for multichannel sampling with unknown offsets using Groebner bases.
1. Write out the equations from (1) describing the samples as a function of the signal coefficients. 2. If necessary, perform a change of variables to convert the equations into a set of polynomial equations. 3. Compute a Groebner basis for the set of polynomial equations using Buchberger's algorithm 4. Use back substitution to compute the offsets and signal parameters from the Groebner basis. 5. If necessary, eliminate solutions that are not valid (e.g. offset values not on the unit circle in the Fourier case).

Example 1 (Polynomial signals): Consider the case where the basis B is given by the functions $_l(t) = t^l$, l = 0,…,L- 1 with two sets of two samples (L = 3, M = 2, and N = 2. Consider the signal parameter vector x = (64 - 24 - 4)^T and the displacements t = (0 1/8)^T. In this case of measurements would be y_0 = (-4.0)^T and y_1 = (-6 6)^T see also Figure 1). We can represent the set of solutions of (31) as the points of the affine variety defined by the set of polynomials:

$$\begin{pmatrix} 0 & 0 & 1 \\ 0.25 & 0.5 & 1 \\ 0.25t_1^2 & 0.5t_1 & 1 \\ (0.5+0.5t_1)^2 & (0.5+0.5t_1) & 1 \end{pmatrix} \begin{pmatrix} _0 \\ _1 \\ _2 \end{pmatrix} = \begin{pmatrix} -4 \\ 0 \\ -6 \\ 6 \end{pmatrix} \qquad (31)$$

We can represent the set of solution of this problem as the points of the affine variety defined by the set of polynomials.

$$a_0 = {}_2 + 4$$
$$a_1 = \frac{1}{4}{}_0 + \frac{1}{2}{}_1 + {}_2$$
$$a_2 = \frac{1}{4}{}_0 t_1^2 + \frac{1}{2}{}_1 t_1 + {}_2 + 6$$
$$a_3 = \frac{1}{4}{}_0 t_1^2 + \frac{1}{2}{}_0 t_1 + \frac{1}{4}{}_0 + \frac{1}{2}{}_1 t_1 + \frac{1}{2}{}_1 + {}_2 - 6$$

in the variables $_0$, $_1$, $_2$ and t_1. We fix the ordering of variables as $_0 \phi\ _1 \phi\ _2 \phi\ t_1$ and we use lex ordering for monomials. At the first step of Buchberger's algorithm, we find that

$$L(a_0, a_1) = 4\,_0 - 2\,_1\,_2 - 4\,_2^2 = (-2\,_1 - 4\,_2)a_0 + 16a_1,$$

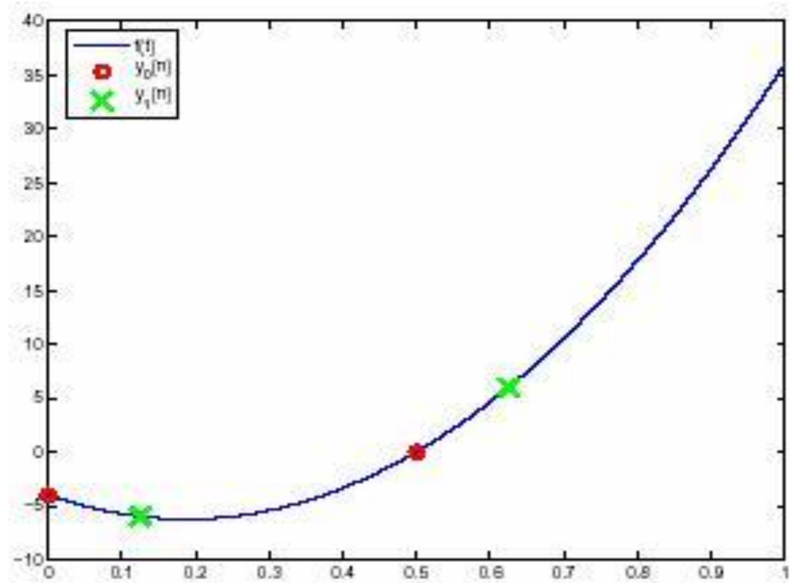

Figure 1. Second degree polynomial signal used in Example 1. The signal f(t) = $64t^2$-24t is sampled with two sets of two samples y_0 = $(-4\ 0)^t$ and y_1 = $(-6\ 6)^t$ with offset t_1 = 1/4.

$$L(a_0,a_2) = {}_0 t_1^2 - \frac{1}{2}{}_1 {}_2 t_1 - {}_2^2 - 6\,{}_2 = (-\frac{1}{2}{}_1 t_1 - {}_2 - 4t_1^2 - 2)a_0 + 4t_1^2 a_1 - 2\,{}_1 t_1 + 16t_1^2 + 8,$$

$$L(a_0,a_3) = -\frac{1}{2}{}_0\,{}_2 t_1 - \frac{1}{4}{}_0\,{}_2 + {}_0 t_1^2 - \frac{1}{2}{}_1\,{}_2 t_1 - \frac{1}{2}{}_1\,{}_2 - {}_2^2 - 6\,{}_2$$
$$= (-\frac{1}{2}{}_0 t_1 - \frac{1}{4}{}_0 - \frac{1}{2}{}_1 t_1 - \frac{1}{2}{}_1 - {}_2 - 4t_1^2 - 8t_1 + 6)a_0 + (4t_1^2 + 8t_1 + 4)a_1$$
$$- 2\,{}_1 t_1^2 - 2\,{}_1 t_1 + 16t_1^2 + 32t_1 - 24,$$

$$L(a_1,a_2) = \frac{1}{8}{}_1 t_1^2 - \frac{1}{8}{}_1 t_1 + \frac{1}{4}{}_2 t_1^2 - \frac{1}{4}{}_2 - \frac{3}{2} = \frac{1}{4}(t_1^2 - 1)a_0 + \frac{1}{8}{}_1 t_1^2 - \frac{1}{8}{}_1 t_1 - t_1^2 - \frac{1}{2}$$

$$L(a_1,a_3) = -\frac{1}{8}{}_0 t_1 - \frac{1}{16}{}_0 + \frac{1}{8}{}_1 t_1^2 - \frac{1}{8}{}_1 t_1 - \frac{1}{8}{}_1 + \frac{1}{4}{}_2 t_1^2 - \frac{1}{4}{}_2 + \frac{3}{2}$$
$$= (\frac{1}{4}t_1^2 + \frac{1}{2}t_1)a_0 + (-\frac{1}{2}t_1 - \frac{1}{4})a_1 + \frac{1}{8}{}_1 t_1^2 + \frac{1}{8}{}_1 t_1 - t_1^2 - 2t_1 + \frac{3}{2},$$

$$L(a_2,a_3) = -\frac{1}{2}{}_0 t_1 - \frac{1}{4}{}_0 - \frac{1}{2}{}_1 + 12 = (2t_1 + 1)a_0 - (2t_1 + 1)a_1 + {}_1 t_1 - 8t_1 + 8$$

Therefore, we add the remainders that are non-zero to the basis:

$$a_4 = \overline{L(a_0,a_2)}^{*} = -2\,{}_1 t_1^2 + 2\,{}_1 t_1 + 16t_1^2 + 8,$$

$$a_5 = \overline{L(a_0,a_3)}^{*} = -2\,{}_1 t_1^2 - 2\,{}_1 t_1 + 16t_1^2 + 32t_1 - 24$$

$$a_6 = \overline{L(a_2,a_3)}^{*} = {}_1 t_1 - 8t_1 + 8$$

The remainders of S(a_1,a_2) and S(a_1,a_3) are not added, because they are the same as polynomials a_4 and a_5, respectively. Following the same procedure, in the second iteration, we find that only S(a_2,a_6) and S(a_4, a_6) give a distinct, non-zero remainder. We add the polynomials

$$a_7 = \overline{L(a_2,a_6)}^{*} = -2\,{}_1 - 48$$

$$a_8 = \overline{L(a_4,a_6)}^{*} = 32t_1 - 8$$

to the basis. In the following iteration all remainders are zero and by Theorem 7.2 we conclude that $a_0, \ldots, a_8$ is a Groebner basis. Applying again Theorem 7.2 we can try to reduce the elements of the basis. In this case, the coefficients a_2, a_3, a_4, a_5, a_6 can be removed and the final basis is given by {a_0, a_1, a_7, a_8}. In order to apply the elimination theorem, we rename the elements of the basis as:

$$G_0 = \frac{1}{4}{}_0 - \frac{1}{2}{}_1 + {}_2,$$

$$G_1 = -2\,{}_1 - 48,$$

$$G_2 = {}_2 + 4,\ G_3 = 32t_1 - 8,$$

The elimination ideals are $I_1 = \langle g_1, g_2, g_3 \rangle$, $I_2 = \langle g_2, g_3 \rangle$

and $I_3 = \langle g_3 \rangle$.The solution of the problem can be obtained by computing the points of the affine variety associated to I_3 and extending it by back substitution to I_2, I_1 and I. We easily find that the unique solution is given by:

$t_1 = 14$, $_2 = -4$, $_1 = -24$, and $_0 = 64$.

The procedure described in the above example can be applied to any multichannel sampling problem in the polynomial space H. For any value of the variables L M, and N, the equations in (6) form a set of polynomial equations and we can therefore compute the parameter values by calculating a Groebner basis for the corresponding ideal. Similarly, the same algorithm can be applied to Fourier series, using the change of variables given in Section 3. This is a very interesting case from a practical point of view, as signals and images are often bandlimited or can be considered to be so.

Conclusions

In this paper, we have presented a method to reconstruct a signal from multiple sets of unregistered, aliased samples using Groebner bases. First, we have shown how multichannel sampling with unknown offsets can be written as a set of polynomial equations. This was shown both for a polynomial signal and for a signal described by its Fourier series. Next, we applied Buchberger's algorithm to compute a Groebner basis for the ideal corresponding to this set of equations.

From a Groebner basis, we can easily derive the unknown signal parameters. We presented an adaptation to our algorithm in the case of noisy measurements. Grobner bases are then computed for critical subsets of the offset polynomials. Finally, some complexity issues are discussed, and more efficient method is presented that computes the linear signal parameters first, such that a Groebner basis has to be computed only for a much smaller set of equations in the unknown offsets. Even after this optimization, such methods memory is requirements. Therefore, we only applied them to one-dimensional signals in our simulations.

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