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## Weak Contractions for Coupled Fixed Point Theorem on G- metric space

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In recent time, fixed point theory has been developed rapidly in partially ordered metric space. Bhaskar and Lakshmikantham (2006) introduced the concept of mixed monotone property. Lakshmikantham and Ćirić (2009) generalized the concept of mixed monotone mapping and proved a common coupled fixed point theorem. In this paper, we find a new type of contractive condition on G- metric spaces also the purpose of this paper is to generalize some recent coupled fixed point theorems in G- metric spaces. We further give some concrete examples in order to support our main theorems. The obtained results generalize those announced by many authors also the main result in this paper is the following coincidence point theorem which generalizes Theorems of Z. Mustafa, W. Shatanawi and M. Batineh (2008)[13] Z. Mustafa, W. Shatanawi and M. Batineh (2009)[14], Z. Mustafa and B. Sims, (2009)[15], H.K. Nashine,(2012)[16] , W. Shatanawi, S. Chauhan, M. Postolache, M., Abbas and S. Radenović (2013)[17], R.Wangkeeree, Bantaojai (2012)[18].

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### INTRODUCTION

Mustafa and Sims (2006) introduced a new notion of generalized metric space denoted G-metric space [11], a generalization of the metric space  $(X, d)$ , to develop and a new fixed point theory for a variety of mappings and to extend known metric space theorems to a more general setting. Subsequently several fixed point results were proven in these spaces [1,2,12,13,14,15].

We present now the necessary definitions and results in G - metric spaces, which will be useful for the rest.

**Definition- 1 [11]** Let  $X$  be a non- empty set and  $G : X \times X \times X \rightarrow R^+$  be a function satisfying the following properties:

- i.  $G(x, y, z) = 0$  if  $x = y = z$
- ii.  $0 < G(x, x, y)$  for all  $x, y \in X$  with  $x \neq y$ ,
- iii.  $G(x, x, y) \leq G(x, y, z)$  for all  $x, y, z \in X$  with  $x \neq y$ ,
- iv.  $G(x, y, z) = G(x, z, y) = G(y, z, x) = \dots$  (symmetry in all three variables),
- v.  $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$  for all  $x, y, z, a \in X$  (rectangle inequality).

Then the function  $G$  is called a generalized metric, or, more specially, a  $G$  - metric on  $X$ , and the pair  $(X, G)$  is called a G- metric space.

**Definition- 2 [11]** Let  $(X, G)$  be a G- metric space, and let  $\{x_n\}$  be a sequence of points of  $X$ , therefore, we say that  $\{x_n\}$  is G- convergent to  $x \in X$  if  $\lim_{n, m \rightarrow +\infty} G(x, x_n, x_m) = 0$ , that is, for any  $\epsilon > 0$ , there exists  $n \in N$

such that  $G(x, x_n, x_m) < \epsilon$  for all  $n, m \geq N$ . We call  $x$  the limit of the sequence and write  $x_n \rightarrow x$  or  $\lim_{n \rightarrow +\infty} x_n = x$ .

**Lemma-3 [11]** Let  $(X, G)$  be a G- metric space. The following statements are equivalent:

- i.  $\{x_n\}$  is G- convergent to  $x$ ,
- ii.  $G(x_n, x_n, x) \rightarrow 0$  as  $n \rightarrow +\infty$ ,
- iii.  $G(x_n, x, x) \rightarrow 0$  as  $n \rightarrow +\infty$ ,
- iv.  $G(x_n, x_m, x) \rightarrow 0$  as  $n, m \rightarrow +\infty$ ,

**Definition-4[11]** Let  $(X, G)$  be a G- metric space. A sequence  $\{x_n\}$  is called a G- Cauchy sequence if, for any  $\epsilon > 0$ , there exists  $n \in N$  such that  $G(x_n, x_m, x_l) < \epsilon$  for all  $n, m, l \geq N$ , that is,  $G(x_n, x_m, x_l) \rightarrow 0$  as  $n, m, l \rightarrow +\infty$ .

**Lemma-5[11]** Let  $(X, G)$  be a G- metric space. The following statements are equivalent:

- i. The sequence  $\{x_n\}$  is G- Cauchy,
- ii. for any  $\epsilon > 0$ , there exists  $n \in N$  such that  $G(x_n, x_m, x_m) < \epsilon$  for all  $m, n \geq N$ .

**Definition-6[11]** A G- metric space  $(X, G)$  is called G- complete if every G- Cauchy sequence is G- convergent in  $(X, G)$ .

Every G- metric on  $X$  defines a metric  $d_G$  on  $X$  given by

$$d_G = G(x, y, y) + G(y, x, x) \text{ for all } x, y \in X.$$

**Lemma -7[11]** If  $X$  is a G - metric space, then  $G(x, y, y) = 2G(y, x, x)$  for all  $x, y \in X$ .

**Lemma-8 [11]** If  $X$  is a G- metric space, then  $G(x, x, y) = G(x, x, z) + G(z, z, y)$  for all  $x, y, z \in X^3$ .

In recent time, fixed point theory has been developed rapidly in partially ordered metric space. Bhaskar and Lakshmikantham (2006)[5] introduced the concept of mixed monotone property. Furthermore, they proved some coupled fixed point theorems for mapping which satisfy the mixed monotone property, and gave a beautiful application in the existence of a solution for a periodic boundary value problem. This concept follows,

**Definition-9** Let  $(X, \leq)$  is a partially ordered set and  $F : X \times X \rightarrow X$ . The mapping  $F$  is said to have the mixed monotone property if  $F$  is nondecreasing monotone in its first argument and is a nonincreasing monotone in its second argument, that is, for any  $x, y \in X$

$$x_1, x_2 \in X, x_1 \leq x_2 \rightarrow F(x_1, y) \leq F(x_2, y) \quad (1.1)$$

and

$$y_1, y_2 \in X, y_1 \leq y_2 \rightarrow F(x, y_1) \geq F(x, y_2) \quad (1.2)$$

Lakshmikantham and Ćirić (2009) [9] generalized the concept of mixed monotone mapping and proved a common coupled fixed point theorem using the following concept of mixed g- monotone mapping.

**Definition-10** Let  $(X, \leq)$  is a partially ordered set and  $F : X \times X \rightarrow X$  and  $g : X \rightarrow X$ . The mapping  $F$  is said to have the mixed g - monotone property if  $F$  is g - nondecreasing monotone in its first argument and is g - nonincreasing monotone in its second argument, that is, for any  $x, y \in X$

$$x_1, x_2 \in X, g(x_1) \leq g(x_2) \rightarrow F(x_1, y) \leq F(x_2, y) \quad (1.3)$$

and

$$y_1, y_2 \in X, g(y_1) \leq g(y_2) \rightarrow F(x, y_1) \geq F(x, y_2) \quad (1.4)$$

Definition – [10] reduces to Definition – [9] when  $g$  is the identity mapping.

**Definition-11** Let  $X$  be a non empty set and  $F : X \times X \rightarrow X$  is said to be continuous if for any two G- convergent sequences  $\{x_n\}$  and  $\{y_n\}$  which converges to  $x$  and  $y$  respectively,  $\{F(x_n, y_n)\}$  is G- convergent to  $F(x, y)$ .

**Definition- 12** Let  $X$  be a non empty set and  $F : X \times X \rightarrow X$  and  $g : X \rightarrow X$  two mappings.  $F$  and  $g$  are commutative if

$$g(F(x, y)) = F(g(x), g(y)), \forall x, y \in X.$$

By using the concept of mixed monotone and mixed g- monotone mapping we prove a coupled fixed point theorem which is generalization of many existing coupled fixed point results on G- metric spaces (2006) [11]. We also give an example in support of our result.

Now in next section we give some previous known results on G- metric space.

## PRELIMINARIES

Denote  $\Phi$  be the set of functions  $\phi : [0, \infty) \rightarrow [0, \infty)$  satisfying the following conditions,

- i.  $\phi$  is continuous and non decreasing,

- ii.  $\phi(t) = 0$  if and only if  $t = 0$ ,
- iii.  $\phi(\alpha t) \leq \alpha \phi(t)$  for  $\alpha \in (0, \infty)$ , and
- iv.  $\phi(t+s) \leq \phi(t) + \phi(s)$  for all  $s, t \in [0, \infty)$ .

Also, let  $\Psi$  be the set of all functions

$$\psi : [0, \infty) \times [0, \infty) \rightarrow [0, \infty)$$

satisfying the condition  $\lim_{t_1 \rightarrow r_1, t_2 \rightarrow r_2} \psi(t_1, t_2) > 0$  for all  $(r_1, r_2) \in [0, \infty) \times [0, \infty)$  with  $r_1 + r_2 > 0$ . For example

- i.  $\psi(t_1, t_2) = k \max\{t_1, t_2\}$  for some  $k \in [0, 1)$ ,
- ii.  $\psi(t_1, t_2) = \alpha t_1^p + \beta t_2^q$  for  $\alpha, \beta, p, q > 0$ , and
- iii.  $\psi(t_1, t_2) = \frac{1-k}{2}(t_1 + t_2)$  for some  $k \in [0, 1)$ .

Choudhury and Maity (2011) [6] gave the first result of coupled fixed point theory. They studied necessary and sufficient conditions for the existence of coupled fixed point in partial ordered G- metric spaces and obtained the following interesting result on G- metric space.

**Theorem-13 [6]** Let  $(X, \leq)$  be a partially ordered set such that  $X$  is a complete G- metric space and  $F: X \times X \rightarrow X$  be a mapping having the mixed monotone property on  $X$ . Suppose that there exists  $k \in [0, 1)$  such that

$$G(F(x, y), F(u, v), F(w, z)) \leq \frac{k}{2} (G(x, u, w) + G(y, v, z)) \quad (2.1)$$

for all  $x, y, z, u, v, w \in X$  for which  $x \geq u \geq w$  and  $y \leq v \leq z$ , where either  $u \neq w$  or  $v \neq z$ . If there exists  $x_0, y_0 \in X$  such that

$$x_0 \leq F(x_0, y_0), \quad y_0 \geq F(y_0, x_0)$$

and either

- i.  $F$  is continuous or
  - ii.  $X$  has the following property:
    - a. if a non decreasing sequence  $\{x_n\}$  such that  $x_n \rightarrow x$  then  $x_n \leq x$  for all  $n$ ,
    - b. if a non increasing sequence  $\{y_n\}$  such that  $y_n \rightarrow y$  then  $y_n \geq y$  for all  $n$ ,
- then  $F$  has a coupled fixed point.

Aydi et al. (2011)[3] generalized this by using the altering distance function and proved the following coupled common fixed point theorem on G- metric space.

**Theorem-14 [3]** Let  $(X, \leq)$  be a partially ordered set such that  $X$  is a complete G- metric space. Suppose that there exist  $\phi \in \Phi$  and  $F: X \times X \rightarrow X$  and  $g: X \rightarrow X$  such that

$$G(F(x, y), F(u, v), F(w, z)) \leq \phi \left( \frac{G(gx, gu, gw) + G(gy, gv, gz)}{2} \right) \quad (2.2)$$

for all  $x, y, z, u, v, w \in X$  for which  $x \geq u \geq w$  and  $y \leq v \leq z$ . Suppose also that  $F$  is continuous and has the mixed monotone property.  $F(X \times X) \subseteq g(X)$  and  $g$  is continuous and commutes with  $F$ . If there exists  $x_0, y_0 \in X$  such that

$$gx_0 \leq F(x_0, y_0) \text{ and } gy_0 \geq F(y_0, x_0)$$

then  $F$  and  $g$  have a coupled coincidence point, that is, there exists  $(x, y) \in X \times X$  such that  $gx = F(x, y)$  and  $gy = F(y, x)$ .

Beside this by using basically concept, Luong and Thuan (2012) [10] presented the following coupled fixed point theorem for nonlinear contractive type mappings having the mixed monotone property in partial ordered G- metric spaces.

**Theorem- 15 [10]** Let  $(X, \leq)$  be a partially ordered set such that  $X$  is a complete G- metric space and  $F: X \times X \rightarrow X$  be a mapping having the mixed monotone property on  $X$ . Suppose that there exists  $\psi \in \Psi$  such that

$$G(F(x, y), F(u, v), F(w, z)) \leq \frac{G(x, u, w) + G(y, v, z)}{2} - \psi(G(x, u, w), G(y, v, z)) \quad (2.3)$$

for all  $x, y, z, u, v, w \in X$  for which  $x \geq u \geq w$  and  $y \leq v \leq z$ , where either  $u \neq w$  or  $v \neq z$ . If there exists  $x_0, y_0 \in X$  such that

$$x_0 \leq F(x_0, y_0) \text{ and } y_0 \geq F(y_0, x_0)$$

and either

- i.  $F$  is continuous or
- ii.  $X$  has the following property:

- a. if a non decreasing sequence  $\{x_n\}$  such that  $x_n \rightarrow x$  then  $x_n \leq x$  for all  $n$ ,  
 b. if a non increasing sequence  $\{y_n\}$  such that  $y_n \rightarrow y$  then  $y_n \geq y$  for all  $n$ ,  
 then  $F$  has a coupled fixed point.

Nashine (2012) [16] introduced a new contractive condition for coupled fixed point theorem in G- metric space and proved the following coupled fixed point result on G- metric spaces.

**Theorem-16 [16]** Let  $(X, G, \leq)$  be a partially ordered G-metric space. Let  $F : X \times X \rightarrow X$  and  $g : X \rightarrow X$  be mappings such that  $F$  has the mixed g-monotone property, and let there exist  $x_0, y_0 \in X$  such that  $gx_0 \leq F(x_0, y_0)$  and  $F(y_0, x_0) \leq gy_0$ . Suppose that there exists  $k \in \left[0, \frac{1}{2}\right)$  such that for all  $x, y, u, v, w, z \in X$ ,

$$G(F(x, y), F(u, v), F(w, z)) + G(F(y, x), F(v, u), F(z, w)) \leq k [G(gx, gu, gw) + G(gy, gv, gz)]$$

Assume the following hypotheses:

- i.  $F(X \times X) \subseteq g(X)$ ,  
 ii.  $g(X)$  is G- complete,  
 iii.  $g$  is G-continuous and commutes with  $F$ .

then  $F$  and  $g$  have a coupled coincidence point, that is, there exists  $(x, y) \in X \times X$  such that  $gx = F(x, y)$  and  $gy = F(y, x)$ .

Karapinar et al. (2012) [7] generalized this result and proved the following common coupled fixed point theorem in G- metric spaces.

**Theorem-17 [7]** Let  $(X, G, \leq)$  be a partially ordered G-metric space. Let  $F : X \times X \rightarrow X$  and  $g : X \rightarrow X$  be mappings such that  $F$  has the mixed g-monotone property, and let there exist  $x_0, y_0 \in X$  such that  $gx_0 \leq F(x_0, y_0)$  and  $F(y_0, x_0) \leq gy_0$ . Suppose that there exists  $k \in [0, 1)$  such that for all  $x, y, u, v, w, z \in X$

$$G(F(x, y), F(u, v), F(w, z)) + G(F(y, x), F(v, u), F(z, w)) \leq k [G(gx, gu, gw) + G(gy, gv, gz)]$$

while

- i.  $F(X \times X) \subseteq g(X)$ ,  
 ii.  $g(X)$  is G- complete, and  
 iii.  $g$  is G-continuous and commutes with  $F$ .

then  $F$  and  $g$  have a coupled coincidence point, that is, there exists  $(x, y) \in X \times X$  such that  $gx = F(x, y)$  and  $gy = F(y, x)$ .

Wangkeeree and Bantaojai (2012) [18] proved the following common coupled fixed point theorems which is generalization of Theorem- 13,

**Theorem- 18 [18]** Let  $(X, \leq)$  be a partially ordered set such that  $X$  is a complete G- metric space and  $F : X \times X \rightarrow X$  and  $g : X \rightarrow X$  be mappings having the mixed g-monotone property on  $X$ . Suppose that there exists  $\psi \in \Psi$  such that

$$G(F(x, y), F(u, v), F(w, z)) \leq [G(gx, gu, gw) + G(gy, gv, gz)] - 2\psi(G(gx, gu, gw), G(gy, gv, gz)) \quad (2.6)$$

for all  $x, y, z, u, v, w \in X$  with  $gx \geq gu \geq gw$  and  $gy \leq gv \leq gz$ . If there exists  $x_0, y_0 \in X$  such that  $gx_0 \leq F(x_0, y_0)$  and  $gy_0 \geq F(y_0, x_0)$

where  $F : X \times X \subseteq g(X)$ ,  $g$  is continuous and commutes with  $F$ , and either

- i.  $F$  is continuous or  
 ii.  $X$  has the following property:  
 a. if a non decreasing sequence  $\{x_n\}$  such that  $x_n \rightarrow x$  then  $x_n \leq x$  for all  $n$ ,  
 b. if a non increasing sequence  $\{y_n\}$  such that  $y_n \rightarrow y$  then  $y_n \geq y$  for all  $n$ ,

then  $F$  and  $g$  have a coupled coincidence point, that is, there exists  $(x, y) \in X \times X$  such that  $g(x) = F(x, y)$  and  $g(y) = F(y, x)$ .

## COUPLED COINCIDENCE POINTS

The main result in this paper is the following coincidence point theorem which generalizes Theorems [13,14,15,16,17,18]

**Theorem-19** Let  $(X, \leq)$  be a partially ordered set such that  $X$  is a complete G- metric space and  $F : X \times X \rightarrow X$  and

$g: X \rightarrow X$  be mappings having the mixed  $g$ -monotone property on  $X$ . Suppose that there exists  $\psi \in \Psi$  and  $\phi \in \Phi$  such that

$$M(x, y, z, u, v, w) \leq \phi \left( \frac{G(gx, gu, gw) + G(gy, gv, gz)}{2} \right) - 2\psi(G(gx, gu, gw), G(gy, gv, gz)) \quad (3.1)$$

where

$$M(x, y, z, u, v, w) = a G(F(x, y), F(u, v), F(w, z)) + b G(F(y, x), F(v, u), F(z, w))$$

for all  $a, b \in (0, \infty)$  and  $x, y, z, u, v, w \in X$  for which  $gx \geq gu \geq gw$  and  $gy \leq gv \leq gz$ . If there exists  $x_0, y_0 \in X$  such that

$$gx_0 \leq F(x_0, y_0) \text{ and } gy_0 \geq F(y_0, x_0)$$

and where  $F: (X \times X) \subseteq g(X)$ ,  $g$  is continuous and commutes with  $F$ , and either

- i.  $F$  is continuous or
- ii.  $X$  has the following property:
  - a. if a non decreasing sequence  $\{x_n\}$  such that  $x_n \rightarrow x$  then  $x_n \leq x$  for all  $n$ ,
  - b. if a non increasing sequence  $\{y_n\}$  such that  $y_n \rightarrow y$  then  $y_n \geq y$  for all  $n$ ,

then  $F$  and  $g$  have a coupled coincidence point, that is, there exists  $(x, y) \in X \times X$  such that

$$g(x) = F(x, y) \text{ and } g(y) = F(y, x).$$

**Proof:** Let  $x_0, y_0 \in X$  satisfy  $gx_0 \leq F(x_0, y_0)$  and  $gy_0 \geq F(y_0, x_0)$ . Since  $F: (X \times X) \subseteq g(X)$ , we can choose  $gx_1, gy_1 \in X$  such that  $gx_1 = F(x_0, y_0)$  and  $gy_1 = F(y_0, x_0)$ . Again since  $F: (X \times X) \subseteq g(X)$ , we can choose  $x_2, y_2 \in X$  such that  $gx_2 = F(x_1, y_1)$  and  $gy_2 = F(y_1, x_1)$ . Continuing this process, we can construct sequences  $\{x_n\}$  and  $\{y_n\}$  in  $X$  such that  $gx_{n+1} = F(x_n, y_n)$  and  $gy_{n+1} = F(y_n, x_n)$

$$\forall n \geq 0. \quad (3.2)$$

Next, we show that

$$g(x_n) \leq g(x_{n+1}) \text{ and } g(y_n) \geq g(y_{n+1}) \quad \forall n \geq 0. \quad (3.3)$$

Since  $g(x_0) \leq F(x_0, y_0) = g(x_1)$  and  $g(y_0) \leq F(y_0, x_0) = g(y_1)$ , therefore, (3.3) holds for  $n = 0$ . Next, suppose that (3.3) holds for some fixed  $n \geq 0$ , that is,

$$g(x_n) \leq g(x_{n+1}) \text{ and } g(y_n) \geq g(y_{n+1}) \quad (3.4)$$

Since  $F$  is the mixed  $g$ -monotone property, from (3.4) and (1.3), imply that

$$F(x_n, y) \leq F(x_{n+1}, y) \text{ and } F(y_{n+1}, x) \leq F(y_n, x) \quad (3.5)$$

for all  $x, y \in X$ . Consequently (3.4) and (1.4) refer that

$$F(y, x_n) \geq F(y, x_{n+1}) \text{ and } F(x, y_{n+1}) \geq F(x, y_n) \quad (3.6)$$

for all  $x, y \in X$ . If we substitute  $y = y_n$  and  $x = x_n$  in (3.5), then we obtain

$$g(x_{n+1}) = F(x_n, y_n) \leq F(x_{n+1}, y_n)$$

and

$$F(y_{n+1}, x_n) \leq F(y_n, x_n) = g(y_{n+1}) \quad (3.7)$$

If we take  $y = y_{n+1}$  and  $x = x_{n+1}$  in (3.6) then

$$F(y_{n+1}, x_n) \geq F(y_{n+1}, x_{n+1}) = g(y_{n+2})$$

and

$$g(x_{n+2}) = F(x_{n+1}, y_{n+1}) \geq F(x_{n+1}, y_n) \quad (3.8)$$

Now, from (3.7) and (3.8), we have

$$g(x_{n+1}) \leq g(x_{n+2}) \text{ and } g(y_{n+1}) \geq g(y_{n+2}). \quad (3.9)$$

By the mathematical induction, we conclude that (3.3) holds for all  $n \geq 0$ . Since

$g(x_n) \leq g(x_{n+1})$  and  $g(y_n) \geq g(y_{n+1})$  for all  $n \geq 0$ , (3.1) implies that

$$M(x_n, x_n, x_{n-1}, y_n, y_n, y_{n-1}) \leq \phi \left( \frac{G(gx_n, gx_n, gx_{n-1}) + G(gy_n, gy_n, gy_{n-1})}{2} \right) - 2\psi(G(gx_n, gx_n, gx_{n-1}), G(gy_n, gy_n, gy_{n-1}))$$

(3.10)

where

$$M(x_n, x_n, x_{n-1}, y_n, y_n, y_{n-1}) = a G(F(x_n, y_n), F(x_n, y_n), F(x_{n-1}, y_{n-1})) + b G(F(y_n, x_n), F(y_n, x_n), F(y_{n-1}, x_{n-1})).$$

Setting

$$w_{n+1}^x = G(gx_{n+1}, gx_{n+1}, gx_n) \quad \forall n \geq 0$$

and

$$w_{n+1}^y = G(gy_{n+1}, gy_{n+1}, gy_n) \quad \forall n \geq 0$$

in (3.10), we obtain

$$a w_{n+1}^x + b w_{n+1}^y \leq \phi \left( \frac{w_n^x + w_n^y}{2} \right) - 2 \psi (w_n^x, w_n^y). \quad (3.11)$$

As  $\psi(t_1, t_2) \geq 0$  for all  $(t_1, t_2) \in [0, \infty) \times [0, \infty)$  we have

$$a w_{n+1}^x + b w_{n+1}^y \leq a w_n^x + b w_n^y, \quad \forall n \geq 0$$

Then the sequence  $\{w_n^x + w_n^y\}$  is decreasing. Therefore, there exists  $w \geq 0$  such that

$$\lim_{n \rightarrow \infty} (a w_n^x + b w_n^y) = \lim_{n \rightarrow \infty} (a G(gx_{n+1}, gx_{n+1}, gx_n) + b G(gy_{n+1}, gy_{n+1}, gy_n))$$

or

$$\lim_{n \rightarrow \infty} (a w_n^x + b w_n^y) = (a + b)w \quad (3.12)$$

Now, we show by contradiction that  $w = 0$ . Suppose that  $w > 0$ . From (3.12) the sequences

$$\{G(gx_{n+1}, gx_{n+1}, gx_n)\} \text{ and } \{G(gy_{n+1}, gy_{n+1}, gy_n)\}$$

have convergent subsequences  $\{G(gx_{n(j)+1}, gx_{n(j)+1}, gx_{n(j)})\}$  and  $\{G(gy_{n(j)+1}, gy_{n(j)+1}, gy_{n(j)})\}$ , respectively.

Assume that

$$\lim_{j \rightarrow \infty} a w_{n(j)}^x = \lim_{j \rightarrow \infty} (G(gx_{n+1}, gx_{n+1}, gx_n)) = a w_1$$

and

$$\lim_{j \rightarrow \infty} b w_{n(j)}^y = \lim_{j \rightarrow \infty} (G(gy_{n+1}, gy_{n+1}, gy_n)) = b w_2$$

which gives that  $aw_1 + bw_2 = (a + b)w$ . From (3.11), we have

$$a w_{n(j)+1}^x + b w_{n(j)+1}^y \leq \phi \left( \frac{w_{n(j)}^x + w_{n(j)}^y}{2} \right) - 2 \psi (w_{n(j)}^x, w_{n(j)}^y). \quad (3.13)$$

Then taking the limit as  $j \rightarrow \infty$  in the above inequality, we obtain

$$(a + b)w \leq \phi \left( \frac{w}{2} \right) - 2 \lim_{j \rightarrow \infty} \psi (w_{n(j)}^x, w_{n(j)}^y) < (a + b)w \quad (3.14)$$

which is contradiction. Thus  $w = 0$ , that is

$$\lim_{n \rightarrow \infty} (a w_n^x + b w_n^y) = 0 \quad (3.15)$$

Next, we show that  $\{g(x_n)\}$  and  $\{g(y_n)\}$  are G- Cauchy sequences. On the contrary, assume that at least one of  $\{g(x_n)\}$  or  $\{g(y_n)\}$  is not a G – Cauchy sequence. By Lemma – 5 there is an  $\epsilon > 0$  for which we can find subsequences  $\{g(x_{n(k)})\}, \{g(x_{m(k)})\}$  of  $\{g(x_n)\}$  and  $\{g(y_{n(k)})\}, \{g(y_{m(k)})\}$  of  $\{g(y_n)\}$  with  $n(k) > m(k) \geq k$  such that

$$G(g(x_{n(k)}), g(x_{n(k)}), g(x_{m(k)})) + G(g(y_{n(k)}), g(y_{n(k)}), g(y_{m(k)})) \geq \epsilon \quad (3.16)$$

Further corresponding to  $m(k)$  we can choose  $n(k)$  in such a way that it is the smallest integer with  $n(k) > m(k) \geq k$  and satisfies (3.16) Then

$$G(g(x_{n(k)-1}), g(x_{n(k)-1}), g(x_{m(k)})) + G(g(y_{n(k)-1}), g(y_{n(k)-1}), g(y_{m(k)})) < \epsilon \quad (3.17)$$

By Lemma -5, we have

$$G(g(x_{n(k)}), g(x_{n(k)}), g(x_{m(k)})) \leq G(g(x_{n(k)}), g(x_{n(k)}), g(x_{n(k)-1})) + G(g(x_{n(k)-1}), g(x_{n(k)-1}), g(x_{m(k)})) \quad (3.18)$$

and

$$G(g(y_{n(k)}), g(y_{n(k)}), g(y_{m(k)})) \leq G(g(y_{n(k)}), g(y_{n(k)}), g(y_{n(k)-1})) + G(g(y_{n(k)-1}), g(y_{n(k)-1}), g(y_{m(k)})) \quad (3.19)$$

from (3.16)—(3.19) we have

$$\begin{aligned} \epsilon &\leq G(g(x_{n(k)}), g(x_{n(k)}), g(x_{m(k)})) + G(g(y_{n(k)}), g(y_{n(k)}), g(y_{m(k)})) \leq G(g(x_{n(k)}), g(x_{n(k)}), g(x_{n(k)-1})) + \\ &G(g(y_{n(k)}), g(y_{n(k)}), g(y_{n(k)-1})) \\ &+ G(g(x_{n(k)-1}), g(x_{n(k)-1}), g(x_{m(k)})) + G(g(y_{n(k)}), g(y_{n(k)}), g(y_{m(k)})) \\ &< G(g(x_{n(k)}), g(x_{n(k)}), g(x_{n(k)-1})) \\ &+ G(g(y_{n(k)}), g(y_{n(k)}), g(y_{n(k)-1})) + \epsilon. \end{aligned}$$

Then letting  $k \rightarrow \infty$  in the above inequality and using (3.15), we have

$$\lim_{k \rightarrow \infty} \left[ G \left( g(x_{n(k)}), g(x_{n(k)}), g(x_{m(k)}) \right) + G \left( g(y_{n(k)}), g(y_{n(k)}), g(y_{m(k)}) \right) \right] = \epsilon. \quad (3.20)$$

By Lemma-7 and Lemma -8 , we have

$$\begin{aligned} aG \left( g(x_{n(k)}), g(x_{n(k)}), g(x_{m(k)}) \right) &\leq aG \left( g(x_{n(k)}), g(x_{n(k)}), g(x_{n(k)+1}) \right) + aG \left( g(x_{n(k)+1}, g(x_{n(k)+1}), g(x_{m(k)})) \right) \leq \\ &2aG \left( g(x_{n(k)+1}, g(x_{n(k)+1}), g(x_{n(k)})) \right) + aG \left( g(x_{n(k)+1}, g(x_{n(k)+1}), g(x_{m(k)+1}) \right) \\ &+ aG \left( g(x_{m(k)+1}, g(x_{m(k)+1}), g(x_{m(k)})) \right) \end{aligned} \quad (3.21)$$

and

$$\begin{aligned} bG \left( g(y_{n(k)}), g(y_{n(k)}), g(y_{m(k)}) \right) &\leq bG \left( g(y_{n(k)}), g(y_{n(k)}), g(y_{n(k)+1}) \right) \\ &+ bG \left( g(y_{m(k)+1}, g(y_{m(k)+1}), g(y_{m(k)})) \right) \end{aligned} \quad (3.22)$$

It follows from (3.21) and (3.22) that is

$$\begin{aligned} aG \left( g(x_{n(k)}), g(x_{n(k)}), g(x_{m(k)}) \right) + bG \left( g(y_{n(k)}), g(y_{n(k)}), g(y_{m(k)}) \right) \\ \leq 2(\{ a w_{n(k)+1}^x + b w_{n(k)+1}^y \}) + (\{ a w_{m(k)+1}^x + b w_{m(k)+1}^y \}) + G \left( g(x_{n(k)+1}, g(x_{n(k)+1}), g(x_{m(k)+1}) \right) + \\ G \left( g(y_{n(k)+1}, g(y_{n(k)+1}), g(y_{m(k)+1}) \right) \end{aligned} \quad (3.23)$$

Since  $n(k) \geq m(k)$ , we get

$$g(x_{n(k)}) \geq g(x_{m(k)}) \text{ and } g(y_{n(k)}) \leq g(y_{m(k)})$$

also from (3.1)

$$\begin{aligned} aG \left( g(x_{n(k)+1}, g(x_{n(k)+1}), g(x_{m(k)+1}) \right) + bG \left( g(y_{n(k)+1}, g(y_{n(k)+1}), g(y_{m(k)+1}) \right) \\ = aG \left( F(x_{n(k)}, y_{n(k)}), F(x_{n(k)}, y_{n(k)}), F(x_{m(k)}, y_{m(k)}) \right) \\ + bG \left( F(y_{n(k)}, x_{n(k)}), F(y_{n(k)}, x_{n(k)}), F(y_{m(k)}, x_{m(k)}) \right) \\ \leq \phi \left( \frac{G(g(x_{n(k)}), g(x_{n(k)}), g(x_{m(k)})) + G(g(y_{n(k)}), g(y_{n(k)}), g(y_{m(k)}))}{2} \right) \\ - \psi \left( \begin{array}{l} G \left( g(x_{n(k)}), g(x_{n(k)}), g(x_{m(k)}) \right), \\ G \left( g(y_{n(k)}), g(y_{n(k)}), g(y_{m(k)}) \right) \end{array} \right) \end{aligned} \quad (3.24)$$

In view of (3.23) and (3.24), we have

$$\begin{aligned} 2(\{ a w_{n(k)+1}^x + b w_{n(k)+1}^y \}) + (\{ a w_{m(k)+1}^x + b w_{m(k)+1}^y \}) \\ \geq aG \left( g(x_{n(k)}), g(x_{n(k)}), g(x_{m(k)}) \right) \\ + bG \left( g(y_{n(k)}), g(y_{n(k)}), g(y_{m(k)}) \right) - aG \left( g(x_{n(k)+1}, g(x_{n(k)+1}), g(x_{m(k)+1}) \right) \\ - bG \left( g(y_{n(k)+1}, g(y_{n(k)+1}), g(y_{m(k)+1}) \right) \\ \geq aG \left( g(x_{n(k)}), g(x_{n(k)}), g(x_{m(k)}) \right) \\ + bG \left( g(y_{n(k)}), g(y_{n(k)}), g(y_{m(k)}) \right) \\ - \phi \left( \frac{G(g(x_{n(k)}), g(x_{n(k)}), g(x_{m(k)})) + G(g(y_{n(k)}), g(y_{n(k)}), g(y_{m(k)}))}{2} \right) \\ + 2\psi \left( \begin{array}{l} G \left( g(x_{n(k)}), g(x_{n(k)}), g(x_{m(k)}) \right), \\ G \left( g(y_{n(k)}), g(y_{n(k)}), g(y_{m(k)}) \right) \end{array} \right) \end{aligned} \quad (3.25)$$

This implies that

$$\begin{aligned} & 2(\{a w_{n(k)+1}^x + b w_{n(k)+1}^y\}) + (\{a w_{m(k)+1}^x + b w_{m(k)+1}^y\}) \\ \geq & 2\psi \left( \begin{array}{c} G(g(x_{n(k)}), g(x_{n(k)}), g(x_{m(k)})) \\ G(g(y_{n(k)}), g(y_{n(k)}), g(y_{m(k)})) \end{array} \right) \end{aligned} \quad (3.26)$$

From (3.20), the sequences  $\{G(g(x_{n(k)}), g(x_{n(k)}), g(x_{m(k)}))\}$  and  $G(g(y_{n(k)}), g(y_{n(k)}), g(y_{m(k)}))$  have subsequences converging to say  $\epsilon_1$  and  $\epsilon_2$  respectively, and  $\epsilon_1 + \epsilon_2 = \epsilon > 0$ . By passing to subsequences, we may assume that

$\lim_{k \rightarrow \infty} G(g(x_{n(k)}), g(x_{n(k)}), g(x_{m(k)})) = \epsilon_1$  and  $\lim_{k \rightarrow \infty} G(g(y_{n(k)}), g(y_{n(k)}), g(y_{m(k)})) = \epsilon_2$ . Taking  $k \rightarrow \infty$  in (3.25) and using (3.26), we have

$$\begin{aligned} 0 &= \lim_{k \rightarrow \infty} \left[ \begin{array}{c} 2(\{a w_{n(k)+1}^x + b w_{n(k)+1}^y\}) \\ + (\{a w_{m(k)+1}^x + b w_{m(k)+1}^y\}) \end{array} \right] \\ &\geq \lim_{k \rightarrow \infty} 2\psi \left( \begin{array}{c} G(g(x_{n(k)}), g(x_{n(k)}), g(x_{m(k)})) \\ G(g(y_{n(k)}), g(y_{n(k)}), g(y_{m(k)})) \end{array} \right) > 0 \end{aligned}$$

which is a contradiction. Therefore  $\{g(x_n)\}$  and  $\{g(y_n)\}$  are  $G$ -Cauchy sequences. By  $G$ -completeness of  $X$ , there exists  $x, y \in X$  such that

$$\lim_{n \rightarrow \infty} g(x_n) = x \text{ and } \lim_{n \rightarrow \infty} g(y_n) = y \quad (3.27)$$

This together with the continuity of  $g$  implies that

$$\lim_{n \rightarrow \infty} g(g(x_n)) = g(x)$$

and

$$\lim_{n \rightarrow \infty} g(g(y_n)) = g(y) \quad (3.28)$$

Now suppose that the assumption (i) holds. From (3.2) and the commutativity of  $F$  and  $g$ , we have  $g(x) = \lim_{n \rightarrow \infty} g(g(x_{n+1})) = \lim_{n \rightarrow \infty} g(F(x_n, y_n)) = \lim_{n \rightarrow \infty} F(g(x_n), g(y_n))$ .  $F(\lim_{(n \rightarrow \infty)g(x_n)}, \lim_{(n \rightarrow \infty)g(y_n)}) = F(x, y)$

Similarly, we have

$$g(y) = \lim_{n \rightarrow \infty} g(g(y_{n+1})) = \lim_{n \rightarrow \infty} g(F(y_n, x_n)) = \lim_{n \rightarrow \infty} F(g(y_n), g(x_n))$$

$$F(\lim_{(n \rightarrow \infty)g(y_n)}, \lim_{(n \rightarrow \infty)g(x_n)}) = F(y, x)$$

Hence,  $(x, y)$  is coupled coincidence point of  $F$  and  $g$ .

Finally suppose that assumption (ii) holds. Since  $\{g(x_n)\}$  is non decreasing satisfying  $g(x_n) \rightarrow x$  and  $\{g(y_n)\}$  is non increasing satisfying  $g(y_n) \rightarrow y$ , we have

$$g(g(x_n)) \leq g(x) \text{ and } g(g(y_n)) \geq g(y), \forall n \geq 0.$$

Using the rectangle inequality and (3.1) we get

$$a G(F(x, y), g(x), g(x)) + b G(F(y, x), g(y), g(y)) \leq$$

$$a G(F(x, y), g(g(x_{n+1})), g(g(x_{n+1}))) + a G(g(g(x_{n+1})), g(g(x)), g(x))$$

$$+ b G(F(y, x), g(g(y_{n+1})), g(g(y_{n+1}))) + b G(g(g(y_{n+1})), g(g(y)), g(y))$$

$$= a G(F(x, y), g(F(x_n, y_n), g(F(x_n, y_n)))) + a G(g(g(x_{n+1})), g(g(x)), g(x))$$

$$+ b G(F(y, x), g(F(y_n, x_n), g(F(y_n, x_n)))) + b G(g(g(y_{n+1})), g(g(y)), g(y))$$

$$\leq \phi \left( \frac{G(g(x), g(g(x_n)), g(g(x_n))) + G(g(y), g(g(y_n)), g(g(y_n)))}{2} \right)$$

$$- \psi \left( \begin{array}{c} G(g(x), g(g(x_n), g(g(x_n)))) \\ G(g(y), g(g(y_n), g(g(y_n)))) \end{array} \right)$$

$$+ G(g(g(x_{n+1})), g(g(x)), g(x)) + G(g(g(y_{n+1})), g(g(y)), g(y)) < 0$$



Letting  $n \rightarrow \infty$  in the above inequality, we obtain that

$$G(F(x, y), g(x), g(x)) + G(F(y, x), g(y), g(y)) = 0$$

which gives that  $G(F(x, y), g(x), g(x)) = G(F(y, x), g(y), g(y)) = 0$ , that is  $F(x, y) = g(x)$  and  $F(y, x) = g(y)$ . Therefore,  $(x, y)$  is a coupled coincidence point of  $F$  and  $g$ . The proof of the theorem is complete.

**Corollary-20** Let  $(X, \leq)$  be a partially ordered set such that  $X$  is a complete  $G$ - metric space and  $F: X \times X \rightarrow X$  be a mapping having the mixed monotone property on  $X$ . Suppose that there exists  $\psi \in \Psi$  and  $\phi \in \Phi$  such that

$$M(x, y, z, u, v, w) \leq \phi \left( \frac{G(x, u, w) + G(y, v, z)}{2} \right) - 2\psi(G(x, u, w), G(y, v, z)) \quad (3.29)$$

where

$$M(x, y, z, u, v, w) = a G(F(x, y), F(u, v), F(w, z)) + b G(F(y, x), F(v, u), F(z, w))$$

for all  $a, b \in (0, \infty)$  and  $x, y, z, u, v, w \in X$  for which  $x \geq u \geq w$  and  $y \leq v \leq z$ . If there exists  $x_0, y_0 \in X$  such that

$$x_0 \leq F(x_0, y_0) \text{ and } y_0 \geq F(y_0, x_0)$$

and either

- i.  $F$  is continuous or
  - ii.  $X$  has the following property:
    - a. if a non decreasing sequence  $\{x_n\}$  such that  $x_n \rightarrow x$  then  $x_n \leq x$  for all  $n$ ,
    - b. if a non increasing sequence  $\{y_n\}$  such that  $y_n \rightarrow y$  then  $y_n \geq y$  for all  $n$ ,
- then  $F$  has a coupled fixed point in  $X$ , that is,  $x = F(x, y)$  and  $y = F(y, x)$ .

**Proof :** Setting  $g(x) = x$  in Theorem-19, then the result follows.

**Theorem-21** Let  $(X, \leq)$  be a partially ordered set such that  $X$  is a complete  $G$ - metric space and  $F: X \times X \rightarrow X$  and  $g: X \rightarrow X$  be mappings having the mixed  $g$ -monotone property on  $X$ . Suppose that there exists  $\psi \in \Psi$  such that

$$M(x, y, z, u, v, w) \leq \frac{G(gx, gu, gw) + G(gy, gv, gz)}{2} - 2\psi(G(gx, gu, gw), G(gy, gv, gz)) \quad (3.30)$$

where

$$M(x, y, z, u, v, w) = a G(F(x, y), F(u, v), F(w, z)) + b G(F(y, x), F(v, u), F(z, w))$$

for all  $a, b \in (0, \infty)$  and  $x, y, z, u, v, w \in X$  for which  $gx \geq gu \geq gw$  and  $gy \leq gv \leq gz$ . If there exists  $x_0, y_0 \in X$  such that

$$gx_0 \leq F(x_0, y_0) \text{ and } gy_0 \geq F(y_0, x_0)$$

and suppose  $F: (X \times X) \subseteq g(X)$ ,  $g$  is continuous and commutes with  $F$ , and also suppose either

- i.  $F$  is continuous or
  - ii.  $X$  has the following property:
    - a. if a non decreasing sequence  $\{x_n\}$  such that  $x_n \rightarrow x$  then  $x_n \leq x$  for all  $n$ ,
    - b. if a non increasing sequence  $\{y_n\}$  such that  $y_n \rightarrow y$  then  $y_n \geq y$  for all  $n$ ,
- then  $F$  and  $g$  have a coupled coincidence point, that is, there exists  $(x, y) \in X \times X$  such that

$$g(x) = F(x, y) \text{ and } g(y) = F(y, x).$$

**Proof:** It is sufficient if we take  $\phi(t) = t$  in Theorem -19 then the result follows.

**Theorem-22** Let  $(X, \leq)$  be a partially ordered set such that  $X$  is a complete  $G$ - metric space and  $F: X \times X \rightarrow X$  and  $g: X \rightarrow X$  be mappings having the mixed  $g$ -monotone property on  $X$ . Suppose that there exists  $\psi \in \Psi$  and  $\phi \in \Phi$  such that

$$M(x, y, z, u, v, w) \leq \phi \left( \frac{G(gx, gu, gw) + G(gy, gv, gz)}{2} \right) - 2\psi(\max \{G(gx, gu, gw), G(gy, gv, gz)\}) \quad (3.31)$$

where

$$M(x, y, z, u, v, w) = a G(F(x, y), F(u, v), F(w, z)) + b G(F(y, x), F(v, u), F(z, w))$$

for all  $a, b \in (0, \infty)$  and  $x, y, z, u, v, w \in X$  for which  $gx \geq gu \geq gw$  and  $gy \leq gv \leq gz$ . If there exists  $x_0, y_0 \in X$  such that

$$gx_0 \leq F(x_0, y_0) \text{ and } gy_0 \geq F(y_0, x_0)$$

and suppose  $F: (X \times X) \subseteq g(X)$ ,  $g$  is continuous and commutes with  $F$ , and also suppose either

- i.  $F$  is continuous or
- ii.  $X$  has the following property:
  - a. if a non decreasing sequence  $\{x_n\}$  such that  $x_n \rightarrow x$  then  $x_n \leq x$  for all  $n$ ,
  - b. if a non increasing sequence  $\{y_n\}$  such that  $y_n \rightarrow y$  then  $y_n \geq y$  for all  $n$ ,

then  $F$  and  $g$  have a coupled coincidence point, that is, there exists  $(x, y) \in X \times X$  such that  $g(x) = F(x, y)$  and  $g(y) = F(y, x)$ .

**Proof:** It is sufficient if we take  $\psi(t_1, t_2) = \max\{t_1, t_2\}$  in Theorem-19, we get the above result.

**Theorem- 23** Let  $(X, \leq)$  be a partially ordered set such that  $X$  is a complete  $G$ - metric space and  $F: X \times X \rightarrow X$  and  $g: X \rightarrow X$  be mappings having the mixed  $g$ -monotone property on  $X$ . Suppose that there exists  $\psi \in \Psi$  and  $\phi \in \Phi$  such that

$$M(x, y, z, u, v, w) \leq \phi \left( \frac{G(gx, gu, gw) + G(gy, gv, gz)}{2} \right) - 2\psi(G(gx, gu, gw), G(gy, gv, gz)) \quad (3.32)$$

where

$$M(x, y, z, u, v, w) = a G(F(x, y), F(u, v), F(w, z)) + b G(F(y, x), F(v, u), F(z, w))$$

for all  $a, b \in (0, \infty)$  and  $x, y, z, u, v, w \in X$  for which  $gx \geq gu \geq gw$  and  $gy \leq gv \leq gz$ . If there exists  $x_0, y_0 \in X$  such that

$$gx_0 \leq F(x_0, y_0) \text{ and } gy_0 \geq F(y_0, x_0)$$

and suppose  $F: (X \times X) \subseteq g(X)$ ,  $g$  is continuous and commutes with  $F$ , and also suppose either

- i.  $F$  is continuous or
- ii.  $X$  has the following property:
  - a. if a non decreasing sequence  $\{x_n\}$  such that  $x_n \rightarrow x$  then  $x_n \leq x$  for all  $n$ ,
  - b. if a non increasing sequence  $\{y_n\}$  such that  $y_n \rightarrow y$  then  $y_n \geq y$  for all  $n$ ,

then  $F$  and  $g$  have a coupled coincidence point, that is, there exists  $(x, y) \in X \times X$  such that

$$g(x) = F(x, y) \text{ and } g(y) = F(y, x).$$

**Proof:** In Theorem – 19, taking  $\psi(t_1, t_2) = \psi(t_1 + t_2)$  for all  $(t_1, t_2) \in [0, \infty)^2$  we get the desired results.

**Theorem-24** In addition to the hypothesis of Theorem – 19, suppose that for all  $(x, y), (x^*, y^*) \in X \times X$ , there exists  $(u, v) \in X \times X$  such that  $(F(u, v), F(v, u))$  is comparable with  $(F(x, y), F(y, x))$  and  $(F(x^*, y^*), F(y^*, x^*))$ . Then  $F$  and  $g$  have a unique coupled common fixed point.

**Proof:** From Theorem- 19, the set of coupled coincidence is non empty. Assume that  $(x, y)$  and  $(x^*, y^*)$  are coupled coincidence points of  $F$  and  $g$ . We shall show that

$$g(x) = g(x^*) \text{ and } g(y) = g(y^*) \quad (3.33)$$

By assumption, there exists  $(u, v) \in X \times X$  such that  $(F(u, v), F(v, u))$  is comparable with  $(F(x, y), F(y, x))$  and  $(F(x^*, y^*), F(y^*, x^*))$ . Putting  $u_0 = u, v_0 = v$  and choosing  $u_1, v_1 \in X$  such that

$$g(u_1) = F(u_0, v_0) \text{ and } g(v_1) = F(v_0, u_0).$$

Then, similarly as in the proof of Theorem – 19, we can inductively define sequences  $\{g(u_n)\}$  and  $\{g(v_n)\}$  in  $X$  by

$$g(u_{n+1}) = F(u_n, v_n) \text{ and } g(v_{n+1}) = F(v_n, u_n),$$

$\forall n \geq 0$

Since  $(F(x^*, y^*), F(y^*, x^*)) = (g(x^*), g(y^*))$  and  $(F(u, v), F(v, u)) = (g(u_1), g(v_1))$  are comparable, so without loss of generality, we may assume that

$$(F(x^*, y^*), F(y^*, x^*)) = (g(x^*), g(y^*)) \leq (F(u, v), F(v, u)) = (g(u_1), g(v_1))$$

and

$$(F(x, y), F(y, x)) = (g(x), g(y)) \leq (F(u, v), F(v, u)) = (g(u_1), g(v_1))$$

This means that

$$g(x^*) \leq g(u_1) \text{ and } g(y^*) \geq g(v_1)$$

and

$$g(x) \leq g(u_1) \text{ and } g(y) \geq g(v_1)$$

Using the fact that  $F$  is a mixed  $g$ -monotone mapping, we can inductively show that

$$g(x^*) \leq g(u_n) \text{ and } g(y^*) \geq g(v_n), \forall n \geq 1$$

and

$$g(x) \leq g(u_n) \text{ and } g(y) \geq g(v_n), \forall n \geq 1$$

Thus from (3.1), we get

$$\begin{aligned} & a G(g(u_{n+1}), g(x), g(x)) + b G(g(v_{n+1}), g(y), g(y)) \\ &= a G(F(u_n, v_n), F(x, y), F(x, y)) + b G(F(v_n, u_n), F(y, x), F(y, x)) \\ &\leq \phi \left( \frac{G(g(u_n), g(x), g(x)) + G(g(v_n), g(y), g(y))}{2} \right) \end{aligned}$$

$$- 2\psi \left( G(g(u_n), g(x), g(x)) + G(g(v_n), g(y), g(y)) \right) \quad (3.34)$$

which implies that

$$aG(g(u_{n+1}), g(x), g(x)) + bG(g(v_{n+1}), g(y), g(y)) \leq aG(g(u_n), g(x), g(x)) + bG(g(v_n), g(y), g(y))$$

that is the sequences  $\{G(g(u_n), g(x), g(x)) + G(g(v_n), g(y), g(y))\}$  is decreasing. Therefore there exists  $\delta \geq 0$  such that

$$\lim_{n \rightarrow \infty} [G(g(u_n), g(x), g(x)) + G(g(v_n), g(y), g(y))] = \delta$$

We shall show that  $\delta = 0$  suppose to the contrary that  $\delta > 0$ . Therefore,  $G(g(u_n), g(x), g(x))$  and  $G(g(v_n), g(y), g(y))$  have subsequences converging to  $\delta_1, \delta_2$  respectively, with

$$\delta_1 + \delta_2 = \delta > 0$$

Taking the limit up to subsequences as  $n \rightarrow \infty$  in (3.34) we have

$$\delta \leq \delta - 2\lim_{n \rightarrow \infty} \psi \left( \begin{matrix} G(g(u_n), g(x), g(x)), \\ G(g(v_n), g(y), g(y)) \end{matrix} \right)$$

$$< \delta$$

which is a contradiction. Thus  $\delta = 0$ , that is

$$\lim_{n \rightarrow \infty} (G(g(u_n), g(x), g(x)) + G(g(v_n), g(y), g(y))) = 0$$

which implies that

$$\lim_{n \rightarrow \infty} G(g(u_n), g(x), g(x)) = \lim_{n \rightarrow \infty} G(g(v_n), g(y), g(y)) = 0 \quad (3.35)$$

Similarly, one can show that

$$\lim_{(n \rightarrow \infty)} G(g(u_n), g(x^*), g(x^*)) = \lim_{n \rightarrow \infty} G(g(v_n), g(y^*), g(y^*)) = 0 \quad (3.36)$$

Therefore, from (3.35), (3.36) and the uniqueness of the limit, we get  $g(x) = g(x^*)$  and  $g(y) = g(y^*)$ . So (3.33) holds. Since  $g(x) = F(x, y)$  and  $g(y) = F(y, x)$ , by commutativity of  $F$  and  $g$  we have

$$g(g(x)) = g(F(x, y)) = F(g(x), g(y)) \text{ and } g(g(y)) = g(F(y, x)) = F(g(y), g(x)) \quad (3.37)$$

Denote  $g(x) = z$  and  $g(y) = w$ , then by (3.37) we get

$$g(z) = F(z, w) \text{ and } g(w) = F(w, z) \quad (3.38)$$

Thus  $(z, w)$  is a coincidence point. Then from (3.33) with  $x^* = z$  and  $y^* = w$ , we have  $g(x) = g(z)$  and  $g(y) = g(w)$ , that is  $g(z) = z$  and  $g(w) = w$ . (3.39)

From (3.38) and (3.39) we get

$$g(z) = F(z, w) = z \text{ and } g(w) = F(w, z) = w \quad (3.40)$$

Then  $(z, w)$  is a coupled common fixed point of  $F$  and  $g$ . To prove the uniqueness, assume that  $(p, q)$  is another coupled fixed point. Then by (3.33) we have

$$g(z) = g(p) = z \text{ and } g(w) = g(q) = q \quad (3.41)$$

This complete the proof of the Theorem.

#### Remark- 25

Some special cases of Theorem 19 yields existing results as detailed below,

In Theorem – 19, if we take following conditions then we get existing results:

- i. If we take  $a = 1, b = 0, \phi(t) = kt$ , where  $k \in (0, 1)$   $g = I_X$  (identity mapping) and  $\psi(t_1, t_2) = 0$  then we get the result of Choudhury and Maity (2011)[6].
- ii. If we take  $a = 1, b = 0$  and  $\psi(t_1, t_2) = 0$  then we get the result of Aydi et al.(2011) [3].
- iii. If we take  $a = 1, b = 1, \phi(t) = 2kt$  for  $k \in [0, \frac{1}{2})$  and  $\psi(t_1, t_2) = 0$  then we get the result of Nashine (2012)[16].
- iv. If we take  $a = 1, b = 1, \phi(t) = 2kt$  for  $k \in [0, 1)$  and  $\psi(t_1, t_2) = 0$  then we get the result of Karapinar et al. (2012) [7].
- v. If we take  $a = 1, b = 0, \phi(t) = 2t$  then we get the result of Wangkeeree and Bantaojai (2012)[18].

vi. If we take  $a = 1, b = 0, g = I_X$  (identity mapping)  $\phi(t) = t$  then we get the result of Luong and Thuan (2012)[10].

**Example-26** Let  $X = \mathbb{R}$ . Define  $G: X \times X \times X \rightarrow [0, \infty)$  by

$$G(x, y, z) = |x - y| + |y - z| + |z - x|$$

$$F(x, y) = 2x - 3y, g(x) = x$$

also  $a = 2, b = 2, \phi(t) = 6t$  and  $\psi(t_1, t_2) = \frac{t_1 + t_2}{4}$ . Then (3.1) indicates that  $(0, 0, 0)$  is a fixed point.

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